Douglass Houghton Workshop, Section 1, Mon 4/2/2012

Worksheet Once More, Unto the Breach, Dear Friends

1. On April first, Dava and Kristy like to play practical jokes on people. Their rule, of course, is that every practical joke done on them demands an equal and opposite retaliation. From year to year they keep a mental "balance sheet" that records how much grief they "owe" or are "owed" by their rivals. Due to various interventions by authority, they have been very good the last few years, and they currently "owe" 100 practical jokes.

   They decide that every year, they will pay off 20% of their "debt", by playing pranks on their friends. At the same time, their friends play 5 pranks on them each year.

   (a) What will the "balance" be at the end of 4/1/2012?

   (b) Fill in the table with the balance B_n at the end of $4/1/(2011+n)$.

n	0	1 (2012)	2 (2013)	3	4	5
B_n	100					

   (c) Find a formula for B_n in terms of n.

   (d) What happens in the long run? (Does the sequence $B_0, B_1, B_2, \ldots$ converge?)

   (e) What if the pranks played happened continuously, instead of once a year. Write a differential equation describing that situation.

Any function which is continuous on $[-\pi, \pi]$ has a **Fourier series**. That is,

$$f(x) = A + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

for some constants $A, a_1, a_2, \ldots, b_1, b_2, \ldots$ Back before spring break we discovered how to compute the constants, namely

$$A = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

2. Let's compute the Fourier series for $f(x) = x^2$.

   (a) Compute A. (A stands for "average".)

   (b) Fill in the table to the right.

   (c) Find the a_n and b_n for $f(x) = x^2$.

n	1	2	3	4	5	6
$\sin(n\pi)$						
$\sin(-n\pi)$						
$\cos(n\pi)$						
$\cos(-n\pi)$						

3. When you bet on a game n times and win with a probability p, the distribution of the number of wins is approximated by the **normal distribution** with pdf

$$\varphi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

Here μ and σ are constants,

$$\begin{aligned} \mu &= \text{mean number of wins} = np \\ \sigma &= \text{standard deviation} = \sqrt{np(1-p)}. \end{aligned}$$

(a) Graph $\varphi(x)$ for 360 games of roulette, which has $p = 9/19$.

(b) Write down an integral for the probability you lose money.

(c) That was ugly. Do a substitution to simplify it as much as possible.

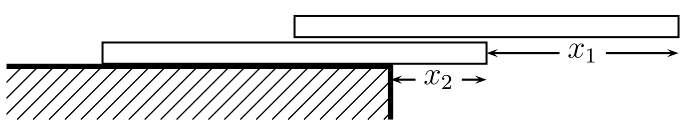
(d) We still can't do that integral. Hmmm. Try using the power series we found for e^x. Can we do it now?

4. Find the full Taylor series for $f(x) = \dfrac{1}{\sqrt{1-4x}}$ about $x = 0$. Also find the interval of convergence.

5. We're interested in how far you can get a playing card to stick out over the edge of a table. We know there must be some limit, but what is it? With two cards, we found

(A) The center of mass of the top card must be over the bottom card, or else it will fall off. The best we can do is take it right to the edge.

(B) The center of mass of both cards must be over the table, or else they will both fall off together. Again, the best we can do is put it right at the edge.



(a) If the cards in the picture are extended as far as possible, find x_1 and x_2.

(b) Put a third card under the first two, flush with the edge of the table. How far can you push the stack out? Hint: find the center of mass of the stack.

6. Over the weekend Janine participated in the UM Dance Marathon for pediatric rehabilitation. She danced for 30 hours, from 10AM Saturday morning until 4PM Sunday afternoon.

As she tired out, she took fewer steps. Assume that each hour she took 7% fewer steps than the hour before. Over the course of the marathon, she took a total of 50,000 steps.

(a) How many steps did Janine take in the first hour? How many in the last hour?

(b) How many did she take on Sunday?