Douglass Houghton Workshop, Section 1, Wed 3/7/2012

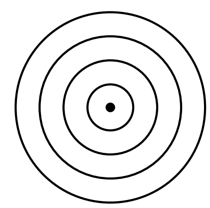
Worksheet Lofty Ambitions

1. Find the probability of winning the "Pass" bet in craps. **Progress:** Last time we calculated the probability that the first roll is an 8 and then we go on to win. We found it to be

$$\begin{aligned} W_4 &= \left(\frac{5}{36}\right)^2 + \left(\frac{5}{36}\right)\left(\frac{25}{36}\right)\left(\frac{5}{36}\right) + \left(\frac{5}{36}\right)\left(\frac{25}{36}\right)^2\left(\frac{5}{36}\right) + \left(\frac{5}{36}\right)\left(\frac{25}{36}\right)^3\left(\frac{5}{36}\right) + \cdots \\ &= \left(\frac{5}{36}\right)^2\left(1 + \left(\frac{25}{36}\right) + \left(\frac{25}{36}\right)^2 + \left(\frac{25}{36}\right)^3 + \cdots\right) \\ &= \left(\frac{5}{36}\right)^2\left(\frac{1}{1-\left(\frac{25}{36}\right)}\right) = \frac{25}{396}. \end{aligned}$$

   Now if we did the same for the other points, and added up the results (together with the probability of a 7 or 11 on the first roll) we'd have the answer.

2. Consider a game of "continuous darts". The board is circular, as you expect, with radius 1. The goal is to get as close to the middle as possible. If a dart lands a distance r from the bullseye, its score is $1 - r$. (So every number between 0 and 1 is a possible score.)



   A novice player throws a dart which lands randomly somewhere on the board. That means that for any region R on the board,

$$\text{Prob(dart lands in } R) = \frac{\text{area of } R}{\text{area of board}}.$$

   Progress: Last time we found that

$$\text{Prob(score} < x) = 1 - (1-x)^2 = 2x - x^2.$$

   Call that function $P(x)$. It's the *Cumulative Distribution Function* (CDF) of the score.

   (a) What's the probability that a score is between x and y?

   (b) Suppose that your score is always rounded up to the nearest multiple of 1/2, so that scores between 0 and 1/2 become 1/2, and scores between 1/2 and 1 become 1. What's your average rounded score?

   (c) What if we round up to the nearest 1/3?

   (d) Find a formula for the average score if you round up to the nearest $1/n$.

3. (This problem appeared on a Winter, 2009 Math 116 exam) The quantity

$$\int_1^\infty \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$$

roughly models the resistance that football-shaped plankton encounter when falling through water. Note that $a = 1$, $b = 2$, and $c = 3$ are constants that describe the dimensions of the plankton. Find a value of M for which

$$\int_1^M \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$$

differs from the original model of resistance by at most 0.001. Hint: make use of the integral and the comparison test.

4. In the course of our probability investigations we learned that if $-1 < t < 1$ then

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \cdots$$

where the dots at the end mean the sum goes on forever.

   (a) Take the derivative of both sides, and see what you get.

   (b) Suppose among a certain group of people, 54% get 1 scoop of ice cream, 32% get 2 scoops, and 14% get 3 scoops. What is the average number of scoops per person?

   (c) One question that a casino might want to answer about a game is how long it is likely to go on. Consider the hard-eight (⚃⚃) bet in craps, for instance, where you win with probability $W = \frac{1}{36}$, lose with probability $L = \frac{10}{36}$, and keep rolling with probability $C = \frac{25}{36}$. What is the probability that the bet is resolved on the first roll? The second roll? The kth roll?

   (d) What is the average number of rolls before the bet is resolved?

5. (Fall, 2005) The world shrimp production can be represented by the differential equation

$$\frac{dP}{dt} = -0.1P(P-7),$$

where t is the number of years since 1982 and $P(t)$ is the quantity of shrimp farmed in the world during year t in hundreds of thousands of metric tons. In 1982 the world shrimp production was 100,000 metric tons.

   (a) Determine all of the equilibrium solutions of the differential equation given above. Classify each as either stable or unstable.

   (b) Sketch a graph of the solution to the given initial value problem. Be sure to indicate clearly on your graph where the solution curve is increasing/decreasing and where it is concave up/concave down. Clearly mark the value of any asymptotes.

   (c) Use Eulers method with $\Delta t = 0.5$ to estimate the world shrimp production in the year 1984 ($t = 2$).

   (d) Is the estimate of world shrimp production in part (5c) bigger or smaller than the exact solution to the initial value problem at $t = 2$? Explain.