Douglass Houghton Workshop, Section 2, Thu 09/21/23

Worksheet Harbinger of Things to Come

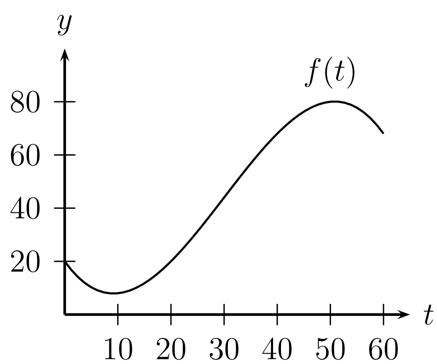
1. Prove that it's possible to make a fair five-sided die. Rules:
   (a) All sides must be flat,
   (b) It must be equally likely to land on all sides, and
   (c) No handles (ala a dreidel).

2. The *power rule for derivatives* says that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. Use the definition of the derivative to prove it for the case where n is a positive integer. Hint: Pascal's triangle.

3. (This problem appeared on a Winter, 2014 Math 115 Exam.) The air in a factory is being filtered so that the quantity of a pollutant, P (in mg/liter), is decreasing exponentially. Suppose t is the time in hours since the factory began filtering the air. Also assume 20% of the pollutant is removed in the first five hours.
   (a) What percentage of the pollutant is left after 10 hours?
   (b) How long is it before the pollution is reduced by 50%?

4. (Fall, 2012) Your pet bird is flying in a straight path toward you and away from you for a minute. After t seconds, she is $f(t)$ feet away from you, where

$$f(t) = \frac{-t(t-20)(t-70)}{500} + 20, \qquad 0 \leq t \leq 60.$$



   (a) Without doing any calculations, determine which is greater: the average velocity of the bird over the entire minute, or her instantaneous velocity after 30 seconds. Explain, referring to the graph.
   (b) Calculate the exact value of the average velocity of the bird over the entire minute.
   (c) Write an explicit expression for the velocity of the bird at time t using the limit definition of velocity. Final answers containing the letter f will receive no credit. Do not evaluate your expression.
   (d) After a minute, you scare the bird, and she flies away at 9 feet/sec. Write a formula for a continuous function $f(t)$ describing the distance between you and the bird for $0 \leq t \leq 180$.

5. (This problem appeared on a Fall, 2010 Math 115 exam.) Before the industrial era, the carbon dioxide (CO_2) level in the air in Ann Arbor was relatively stable with small seasonal fluctuations caused by plants absorbing CO_2 and producing oxygen in its place. Typically, on March 1, the CO_2 concentration reached a high of 270 parts per million (ppm), and on September 1, the concentration was at a low of 262 ppm. Let $G(t)$ be the CO_2 level t months after January 1.

   (a) Assuming that $G(t)$ is periodic and sinusoidal, sketch a neat, well-labeled graph of G with $t = 0$ corresponding to January 1.

   (b) Determine an explicit expression for G, corresponding to your sinusoidal graph above.

6. (Winter, 2010) Suppose $W(h)$ is an invertible function which tells us how many gallons of water an h-foot tall oak tree uses on a hot summer day.

   (a) Give practical interpretations of $W(50)$ and $W^{-1}(40)$.

   (b) Suppose that an average oak tree is A feet tall and uses G gallons of water on a hot summer day. Answer in terms of W, A, and G:

      i. A farmer has 25 oak trees, and each one is 10 feet taller than an average oak tree. How much water will they use on a hot summer day?

      ii. The farmer also has some oak trees which use 5 fewer gallons of water on a hot summer day than an average oak tree does. How tall are his trees?

7. (This problem appeared on a Winter, 2007 Math 115 Exam) Cosmologists, through a technique best described as hocus pocus, measure a quantity $T(t)$, the temperature of the universe in degrees Kelvin (K), where t is in gigayears (Gyr) after the Big Bang. Suppose that, currently, $t = 13.6$, $T(13.6) = 2.4$, and $T'(13.6) = -12$.

   [Note: A gigayear is 1 billion years, and the Kelvin temperature scale is an absolute temperature scale where the lowest possible temperature is defined as being zero Kelvin.]

   (a) For each of the following statements, state whether you agree or disagree with the conclusion and justify your reasoning.

      i. In the next billion years, the temperature of the universe will drop by approximately 12 degrees Kelvin.

      ii. In the next year, the temperature of the universe will drop by approximately $\frac{12}{1{,}000{,}000{,}000}$ degrees Kelvin.

   (b) Assume $T(t)$ is decreasing and does not change concavity on the domain $[13.6, \infty)$. Do you expect $T(t)$ to be concave up or concave down on the domain $[13.6, \infty)$? Justify your answer using physical reasoning.

8. (This problem appeared on a Winter, 2014 Math 115 exam) Find all vertical and horizontal asymptotes of the graph of

$$g(x) = \frac{k(x-a)(x-b)}{(x-a)(x-c)^2}$$

   where a, b, c, and k are constants with $a < b < c < k \neq 0$.