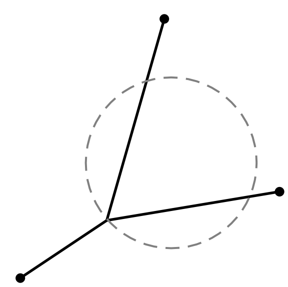
**Douglass Houghton Workshop, Section 2, Tue 11/26/19**

# Worksheet Quoth the Raven, "Nevermore"

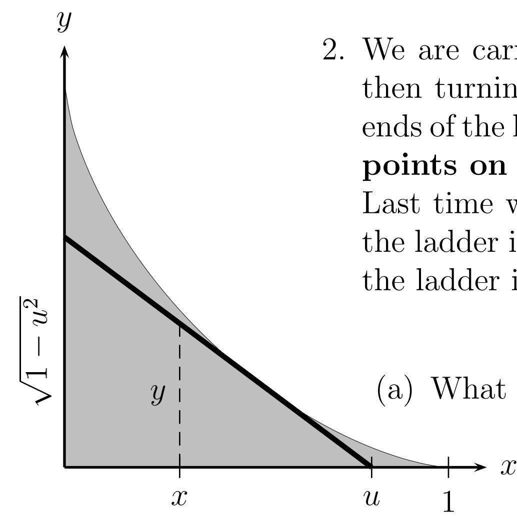
1. Shortest Network. Last time we used calculus to show that a $\Lambda$-shaped network can be improved if the vertex angle is less than 120°.

   (a) Prove that the network to the right is NOT minimizing. You don't need to find the optimal network, just prove that this one can be improved. Hint: consider the portion of the network that is inside the circle.

   (b) What allowed that trick to work? Phrase your answer like this: "Any network which contains ______ can be improved."

   (c) Put it all together, and explain where the soap puts the roundabout.




2. We are carrying a ladder of length 1 down a hallway, and then turning it to get around a corner, always keeping the ends of the ladder against the walls. The question is: **Which points on the floor does the ladder pass over?**
   Last time we found that if $0 \le x \le u \le 1$ and the base of the ladder is at $(u, 0)$, then the distance from $(x, 0)$ north to the ladder is
   $$y = \frac{u-x}{u}\sqrt{1-u^2}.$$

   (a) What value of $u$ maximizes $y$? (Keep $x$ fixed!)

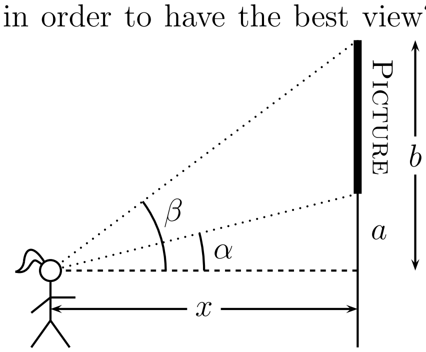
   (b) So for a fixed $x$, what is the maximum value of $y$, as the ladder moves?

   (c) You have found a formula for the curve at the top of the region we want. Simplify until it's beautiful. (This is the best part, so don't stop until it's truly wondrous.)

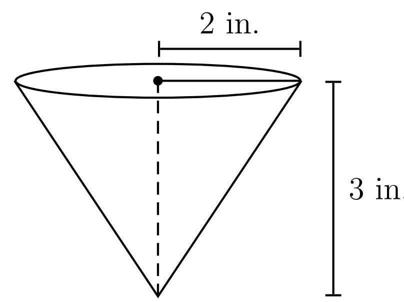
3. Suppose a picture is mounted on the wall. Its bottom is $a$ feet above eye level, and its top is $b$ feet above eye level. If you stand far away from the wall, you can't see the picture well. But if you stand close to the wall, you can't see well either! So the question is: how far from the wall should you stand in order to have the best view?

   (a) Let $\alpha$ and $\beta$ be the angles between eye level and the bottom and top of the picture, as shown. $x$ is your distance from the wall. Find $\alpha$ and $\beta$ in terms of $x$, $a$, and $b$.

   (b) $\beta - \alpha$ is the angle that the picture takes up in your field of vision. So find the value of $x$ that maximizes $\beta - \alpha$.



4. (This problem appeared on the Fall, 2008 Math 115 Final Exam) Suppose that you are brewing coffee and that hot water is passing through a special, cone-shaped filter. Assume that the height of the conic filter is 3 in. and that the radius of the base of the cone is 2 in. If the water is flowing out of the bottom of the filter at a rate of 1.5 $\text{in}^3/\text{min}$ when the remaining water in the filter is 2 in. deep, how fast is the depth of the water changing at that instant?




5. (This problem appeared on the Winter, 2015 Math 115 Final Exam) For nonzero constants $a$ and $b$ with $b > 0$, consider the family of functions given by

$$f(x) = e^{ax} - bx.$$

   (a) Suppose the values of $a$ and $b$ are such that $f(x)$ has at least one critical point. For the domain $(-\infty, \infty)$, find all critical points of $f(x)$, all values of $x$ at which $f(x)$ has a local extremum, and all values of $x$ at which $f(x)$ has an inflection point. (Note that your answer(s) may include the constants $a$ and/or $b$.)

   (b) Which of the following conditions on the constant $a$ guarantee(s) that $f(x)$ has at least one critical point in its domain $(-\infty, \infty)$?

   (i) $a < 0$ (ii) $0 < a < b$ (iii) $b < a$

   (c) Find exact values of $a$ and $b$ so that $f(x)$ has a critical point at $(1, 0)$.

6. Here is the graph of the *derivative* of the continuous function $M(x)$. Using the fact that $M(-4) = -2$, sketch the graph of $M(x)$. Give the coordinates of all critical points, inflection points, and endpoints.