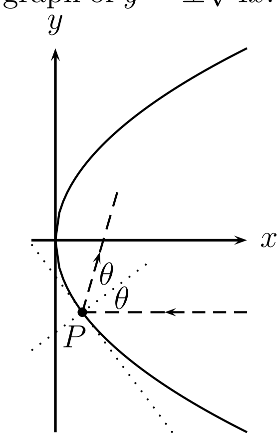
Douglass Houghton Workshop, Section 2, Tue 10/22/19

Worksheet Koala

1. Last time we thought about a parabolic mirror in the shape of the graph of $y = \pm\sqrt{4x}$.

   So far we've found:

   - A light ray $y = -b$ hits the mirror at $P = (b^2/4, -b)$.
   - The slope of the tangent at that point is $-2/b$.
   - The normal line at the same point has slope $b/2$.
   - When a line makes an angle θ with the x-axis, it has slope $\tan\theta$.
   - So if we call the angle between the normal line and the horizontal θ, then $\theta = \tan^{-1}(b/2)$.

   (a) Draw the picture on the board.

   (b) To the ray, the mirror looks flat, just like the tangent line. Draw the reflected ray. What angle does it make with the x-axis?

   (c) We know that $\sin 2x = 2\sin x \cos x$ and $\cos 2x = \cos^2 x - \sin^2 x$. Use those to find a formula for $\tan 2x$ in terms of $\tan x$.

   (d) What is the slope of the reflected ray?

   (e) Write an equation for the reflected ray.

   (f) Where does the reflected ray intersect the x-axis? What is surprising about this answer?

   (g) Graph several rays, with their reflections.

   (h) What's cool about this type of mirror?

2. Let's see if we can prove that the derivative of $\sin(x)$ is $\cos(x)$. Remember that last time we showed that
$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$$

   (a) Show that $\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta} = 0$. Hint: Multiply the top and bottom by $1+\cos(\theta)$, and simplify.

   (b) Write down the definition of the derivative of $\sin(x)$ at $x = a$.

   (c) Use a trig identity to write $\sin(a+h)$ in terms of sines and cosines of a and h.

   (d) Now use the two limits we know (the one from last time and the one in part (a)) to simplify the derivative.

3. (An old team homework problem.) Let $f(x) = x^2 - 2x + 13$ and $g(x) = -x^2 - 2x - 5$.

   (a) Draw $y = f(x)$ and $y = g(x)$ on the same set of axes. How many lines are tangent to both graphs?

   (b) Find the equations of those lines.

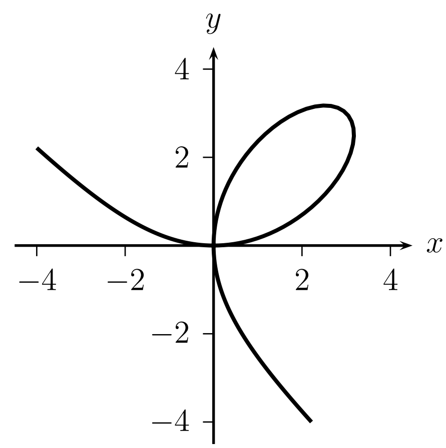
4. (This problem appeared on a Winter, 2005 Math 115 Exam) An example of Descartes' folium, shown in the picture to the right, is given by $x^3 + y^3 = 6xy$.

   (a) Show that the point $(3, 3)$ is on the graph.

   (b) Find the equation of the tangent to the graph at the point $(3, 3)$.

   (c) For what value(s) of x will the tangent to this curve be horizontal? [You do not need to solve for both x and y—just show x in terms of y.]

   (d) (Added for DHSP) Oh heck, go ahead and find the point(s).



5. Let $f^{(n)}(x)$ denote the nth derivative of f. If $f(x) = e^{-2x}$, find $f^{(531)}(x)$. Is $f^{(531)}(x)$ increasing or decreasing? Concave up or concave down? Try graphing $f^{(531)}$ without your calculator, then check with the calculator.

6. Molecules absorb far-infrared radiation because its excites their rotation. The absorption coefficient a of a given liquid varies with the frequency ω of the radiation according to

$$a(\omega) = \frac{10}{\omega^2 - 2c\omega + 125}$$

   where c is some constant $(0 \leq c \leq 11)$.

   (a) For what value of the frequency ω is the absorption a maximum?

   (b) Graph $a(\omega)$ for $c = 11$. How would you describe the shape of this graph?

   [Note: with appropriate parameters this function describes the shapes of the lines in many kinds of spectroscopy].

7. (This problem appeared on a Winter, 2004 Math 115 exam. Really!) While exploring an exotic spring break location, you discover a colony of geese who lay golden eggs. You bring 20 geese back with you. Suppose each goose can lay 294 golden eggs per year. You decide maybe 20 geese isn't enough, so you consider getting some more of these magical creatures. However, for each extra goose you bring home there are less resources for all the geese. Therefore, for each new goose the amount of eggs produced will decrease by 7 eggs per goose per year. How many more geese should you bring back if you want to maximize the number of golden eggs per year laid?