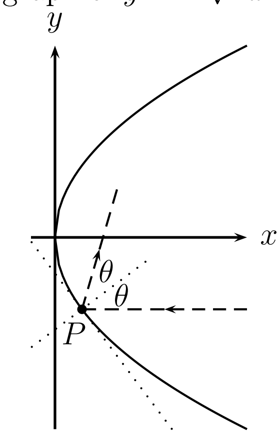
Douglass Houghton Workshop, Section 1, Wed 10/24/18

Worksheet Joy is not in things; it is in us

1. Last time we thought about a parabolic mirror in the shape of the graph of $y = \pm\sqrt{4x}$.

   So far we've found:

   - A light ray $y = -b$ hits the mirror at $P = (b^2/4, -b)$.
   - The slope of the tangent at that point is $-2/b$.
   - The normal line at the same point has slope $b/2$.
   - When a line makes an angle θ with the x-axis, it has slope $\tan\theta$.
   - So if we call the angle between the normal line and the horizontal θ, then $\theta = \tan^{-1}(b/2)$.



   (a) Draw the picture on the board.

   (b) To the ray, the mirror looks flat, just like the tangent line. Draw the reflected ray. What angle does it make with the x-axis?

   (c) We know that $\sin 2x = 2\sin x\cos x$ and $\cos 2x = \cos^2 x - \sin^2 x$. Use those to find a formula for $\tan 2x$ in terms of $\tan x$.

   (d) What is the slope of the reflected ray?

   (e) Write an equation for the reflected ray.

   (f) Where does the reflected ray intersect the x-axis? What is surprising about this answer?

   (g) Graph several rays, with their reflections.

   (h) What's cool about this type of mirror?

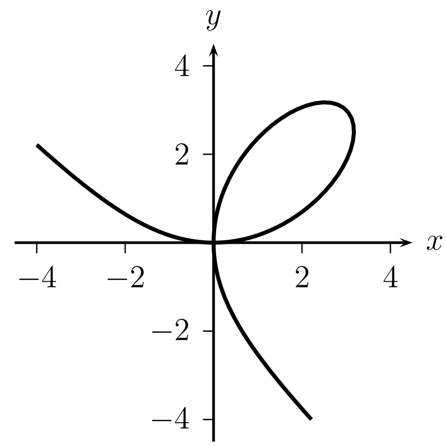
2. (This problem appeared on a Fall, 2006 Math 115 exam) The Flux F, in millilitres per second, measures how fast blood flows along a blood vessel. Poiseuille's Law states that the flux is proportional to the fourth power of the radius, R, of the blood vessel, measured in millimeters. In other words $F = kR^4$ for some positive constant k.

   (a) Find a linear approximation for F as a function of R near $R = 0.5$. (Leave your answer in terms of k).

   (b) A partially clogged artery can be expanded by an operation called an angioplasty, which widens the artery to increase the flow of blood. If the initial radius of the artery was 0.5 mm, use your approximation from part (a) to approximate the flux when the radius is increased by 0.1 mm.

   (c) Is the answer you found in part (b) an under- or over-approximation? Justify your answer.

3. (This problem appeared on a Winter, 2005 Math 115 Exam) An example of Descartes' folium, shown in the picture to the right, is given by $x^3 + y^3 = 6xy$.



   (a) Show that the point $(3, 3)$ is on the graph.

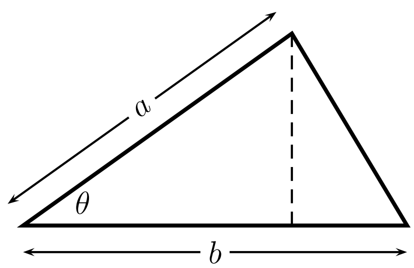
   (b) Find the equation of the tangent to the graph at the point $(3, 3)$.

   (c) For what value(s) of x will the tangent to this curve be horizontal? [You do not need to solve for both x and y—just show x in terms of y.]

   (d) (Added for DHSP) Oh heck, go ahead and find the point(s).

4. (This problem appeared on a Fall, 2008 Math 115 exam) Determine a and b for the function of the form $y = f(t) = at^2 + b/t$, with a local minimum at $(1, 12)$.

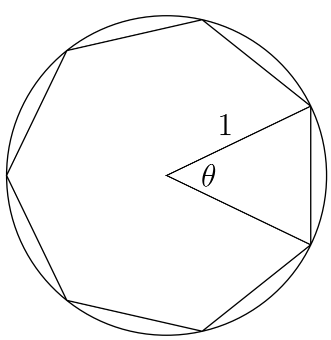
5. Let's find a new formula for the area of a triangle. We know AREA $= \frac{1}{2}$BASE$\times$HEIGHT. But suppose we know two sides a and b of a triangle and the angle θ between them.



   (a) If we consider b as the base of this triangle, then what's the height in terms of a and θ?

   (b) Give a formula for the area of the triangle in terms of a, b, and θ.

6. (This problem explains why $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, but only when θ is measured in radians.) Consider a regular n-sided polygon inscribed in a circle of radius 1.



   (a) Let A_n be the area of the polygon. What does A_n approach as n gets large? $\lim_{n\to\infty} A_n = \boxed{}$

   (b) We can compute A_n by dividing the polygon up into triangles which have a vertex at the center. Let θ be the vertex angle (in radians). What is θ in terms of n?

   (c) What happens to θ as n gets large?

   (d) What is the area of one of the triangles, in terms of θ?

   (e) What is A_n in terms of θ?

   (f) Substitute into the equation from part (a) so that it includes θ's but not n's. Simplify it as much as you can. Hint: $\sin(2x) = 2\sin(x)\cos(x)$.

   (g) What would change if we measured θ in degrees instead of radians?