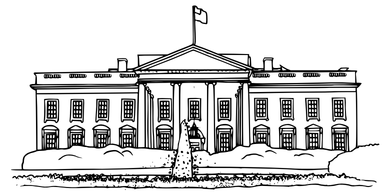
Douglass Houghton Workshop, Section 1, Mon 10/22/18

Worksheet If Wishes Were Changes, We'd All Live in Roses



1. We still have this $1/z$ scale model of the White House, which we plan on blowing up. We want to decide what speed to run the film at, so that when we slow it down to 24 frames per second, we get a realistic explosion.

   (a) Last time we showed that an object will fall $16t^2$ feet in t seconds. So how long does it take for an object to fall off the real white house, which is H feet tall? How many frames will that be, if we film it at 24 frames per second and show it at the same speed?

   (b) How long does it take an object to fall off the top of the model?

   (c) How many frames per second should you film to get the right number of frames to make it look like the model is full-sized?

2. Consider a mirror in the shape of the graph of $y = \pm\sqrt{4x}$.

   (a) Draw the mirror (make it big). What shape is it?

   (b) Draw a light ray travelling leftward along the line $y = -b$, where b is some positive number (making $-b$ negative). At what point P does the ray hit the mirror?

   (c) Find, in terms of b, the slope of the tangent to the mirror at P.

   (d) The *normal* to a curve at a point is the line through that point which is perpendicular to the tangent line. Find the slope of the normal to the mirror at P, and draw both the normal and tangent lines on your graph.

   (e) Suppose a line makes an angle θ with the x-axis. What is the slope of the line?

   (f) Let θ be the angle the normal to the mirror at P makes with the light ray $y = -b$. Can you write θ in terms of b? Hint: Use (2d) and (2e).

   To be continued...

3. (This problem appeared on a Winter 2007 Math 115 exam) Suppose f and g are differentiable functions with values given by the table below.

   (a) If $h(x) = f(x)g(x)$, find $h'(3)$.

   (b) If $j(x) = \dfrac{(g(x))^3}{f(x)}$, find $j'(1)$.

   (c) If $d(x) = x \ln\left(e^{f(x)}\right)$, find $d'(3)$.

   (d) If $t(x) = \cos(g(x))$, find $t'(1)$.

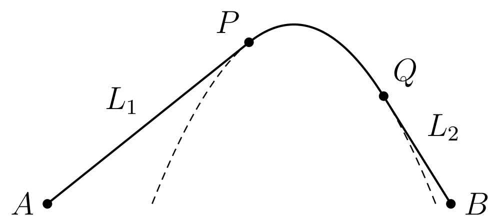
x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	2	9	-3	7
3	4	11	15	-19

(d) If $t(x) = \cos(g(x))$, find $t'(1)$.

4. Suppose you are asked to design the first ascent and drop for a new roller coaster at Cedar Point. (You get to name it, too!) By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop -1.6. You decide to connect these two straight inclines $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola $y = f(x) = ax^2 + bx + c$, where x and $f(x)$ are measured in feet. For the track to be smooth there can't be abrupt changes in direction, so you want the linear segments L_1 and L_2 to be tangent to the parabola at the transition points P and Q. To simplify the equations, you decide to put the origin at P.

   (a) Name your coaster.

   (b) Suppose the horizontal distance between P and Q is 100 feet. Write equations in a, b, and c that will ensure that the track is smooth at the transition points.



   (c) Solve the equations in (4b) for a, b, and c to find a formula for $f(x)$.

   (d) Plot L_1, f, and L_2 to verify graphically that the transitions are smooth.

   (e) Find the difference in elevation between P and Q.

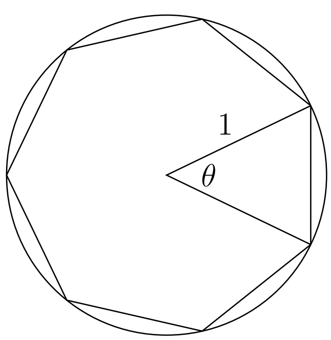
   (f) Suppose the base of the hill (the distance from A to B in the picture) is 300 feet long. How high is the hill?

5. (An old team homework problem.) Let $f(x) = x^2 - 2x + 13$ and $g(x) = -x^2 - 2x - 5$.

   (a) Draw $y = f(x)$ and $y = g(x)$ on the same set of axes. How many lines are tangent to both graphs?

   (b) Find the equations of those lines.

6. (This problem explains why $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, but only when θ is measured in radians.) Consider a regular n-sided polygon inscribed in a circle of radius 1.



   (a) Let A_n be the area of the polygon. What does A_n approach as n gets large? $\lim_{n\to\infty} A_n = \square$

   (b) We can compute A_n by dividing the polygon up into triangles which have a vertex at the center. Let θ be the vertex angle (in radians). What is θ in terms of n?

   (c) What happens to θ as n gets large?

   (d) What is the area of one of the triangles, in terms of θ?

   (e) What is A_n in terms of θ?

   (f) Substitute into the equation from part (a) so that it includes θ's but not n's. Simplify it as much as you can. Hint: $\sin(2x) = 2\sin(x)\cos(x)$.

   (g) What would change if we measured θ in degrees instead of radians?