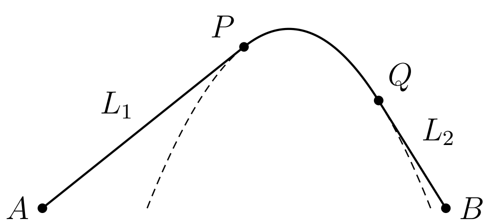
Douglass Houghton Workshop, Section 1, Wed 10/17/18

Worksheet How Noble in Reason, How Infinite in Faculty!

1. Last time we found a way to number the rooms of an infinite decker bus, so that everyone gets a room and no two people get the same room. You can think of that as a function f that takes two natural numbers (say, x and y) as input and produces one (the hotel room) as output. What if you have a 3D bus? That is, every seat has three coordinates (you can call them x, y, and z), and you need to produce the hotel room as output. Hint: Use the function f twice.

1	3	6	10	15	⋯
2	5	9	14		
4	8	13			
7	12				
11					
⋮					

2. Suppose you are asked to design the first ascent and drop for a new roller coaster at Cedar Point. (You get to name it, too!) By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop -1.6. You decide to connect these two straight inclines $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola $y = f(x) = ax^2 + bx + c$, where x and $f(x)$ are measured in feet. For the track to be smooth there can't be abrupt changes in direction, so you want the linear segments L_1 and L_2 to be tangent to the parabola at the transition points P and Q. To simplify the equations, you decide to put the origin at P.



   (a) Name your coaster.

   (b) Suppose the horizontal distance between P and Q is 100 feet. Write equations in a, b, and c that will ensure that the track is smooth at the transition points.

   (c) Solve the equations in (2b) for a, b, and c to find a formula for $f(x)$.

   (d) Plot L_1, f, and L_2 to verify graphically that the transitions are smooth.

   (e) Find the difference in elevation between P and Q.

   (f) Suppose the base of the hill (the distance from A to B in the picture) is 300 feet long. How high is the hill?

3. As we know, Tommy is an excellent cornhole player. But one day while playing cornhole next to a cliff, one beanbag falls over the edge.

   (a) The beanbag's downward acceleration due to gravity is $-32\,\text{ft/sec}^2$. So what is its velocity t seconds after it is dropped? (Keep in mind that the derivative of velocity is acceleration, and the initial velocity is 0.)

   (b) How far has the beanbag fallen after t seconds?

   (c) Suppose the beanbag takes 3.5 seconds to hit the bottom. How high are the cliffs?

   (d) Explain a general method for finding the height of any cliff.

4. Suppose you construct a $1/z$ scale model of the White House, in order to film it blowing up. You will show the film at 24 frames per second. How many frames per second should you *film* so that when you slow the speed down, things will fall at believable speeds?

5. Let r be the radius of a sphere, and let $V = f(r)$ be its volume.

   (a) Find a formula for $f(r)$.

   (b) Find $f'(r)$. Do you recognize this formula? It's on the first page of your book.

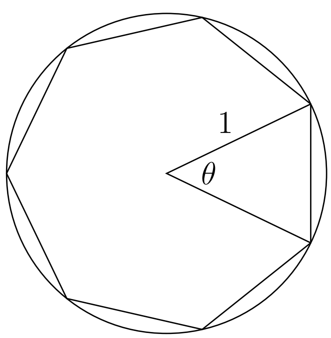
   (c) Suppose a spherical balloon has radius 100 mm, and you pump it up so its radius is a small amount Δr bigger. What would happen to the volume? Answer in two ways:

      i. By interpreting the derivative of volume, and

      ii. By imagining covering the baloon with tinfoil of thickness Δr.

   (d) Now explain the strange coincidence you found in (5b).

6. (This problem explains why $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, but only when θ is measured in radians.) Consider a regular n-sided polygon inscribed in a circle of radius 1.



   (a) Let A_n be the area of the polygon. What does A_n approach as n gets large? $\lim_{n \to \infty} A_n = \square$

   (b) We can compute A_n by dividing the polygon up into triangles which have a vertex at the center. Let θ be the vertex angle (in radians). What is θ in terms of n?

   (c) What happens to θ as n gets large?

   (d) What is the area of one of the triangles, in terms of θ?

   (e) What is A_n in terms of θ?

   (f) Substitute into the equation from part (a) so that it includes θ's but not n's. Simplify it as much as you can. Hint: $\sin(2x) = 2\sin(x)\cos(x)$.

   (g) What would change if we measured θ in degrees instead of radians?