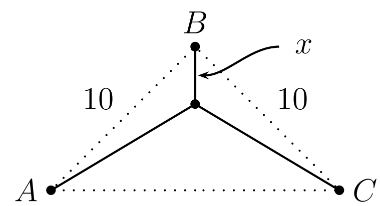
Douglass Houghton Workshop, Section 2, Tue 11/22/16

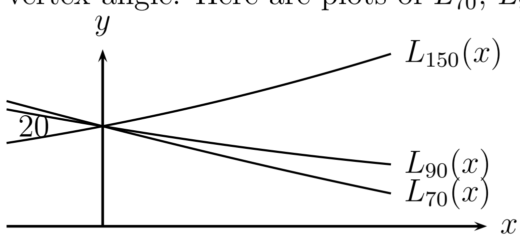
Worksheet Once Upon an Integral

1. Shortest Network. We found a formula for the length of a Y-shaped network connecting three cities at the corners of an isoceles triangle with vertex angle θ:

$$L_\theta(x) = x + 2\sqrt{x^2 - 20\cos(\theta/2)x + 100}$$



Then we noticed that sometimes we could make a Y that improves upon the corresponding V-shaped network, but sometimes we couldn't. The distinction seemed to depend on the vertex angle. Here are plots of L_{70}, L_{90}, and L_{150}, near $x = 0$:



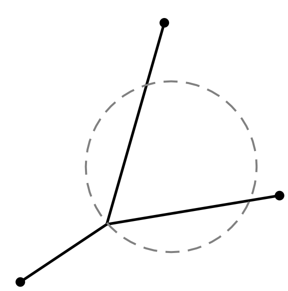
Since $x = 0$ represents the V, it looks like we can tell whether the V can be improved by looking at whether $L_\theta(x)$ is increasing or decreasing at $x = 0$. So figure out which Vs can be improved.

2. Of course, so far we've only considered isosceles triangles. We need to generalize to all possible placements of cities.

   (a) Prove that the network to the right is NOT minimizing. You don't need to find the optimal network, just prove that this one can be improved. Hint: consider improving the portion of the network that is inside the circle.

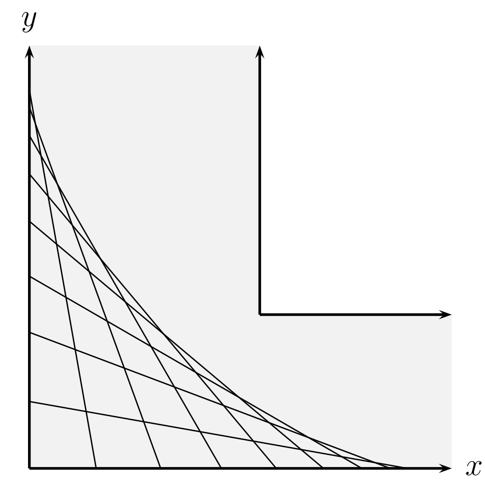
   (b) What allowed that trick to work? Phrase your answer like this: "Any network which contains ________ can be improved."

   (c) Put it all together, and explain where the soap puts the roundabout.



3. (Adapted from a Fall, 2005 Math 115 Final Exam) On Christmas Eve, Jack makes cookies for Santa. Addison, flying her plane, is trying to beat Santa to the cookies. Assume that Santa is directly North of the house (therefore traveling due South) while Addison is directly East of the house (traveling due West—also flying, so as to try to get ahead of Santa). Assume that both Santa and Addison are flying at the same altitude. Santa is moving at 30 miles per hour, and Addison is going 28 miles per hour. How fast is the distance between them changing when Santa is 120 miles from Jack's house and Addison is 160 miles from the house?

4. Suppose we are carrying a ladder down a hallway, and then turning it to get around a corner, always keeping the ends of the ladder against the walls. The question is: **Which points on the floor does the ladder pass over?** Let's assume the ladder has length 1. In the picture, the gray area is the hallway and the fine lines are the ladder in different positions.



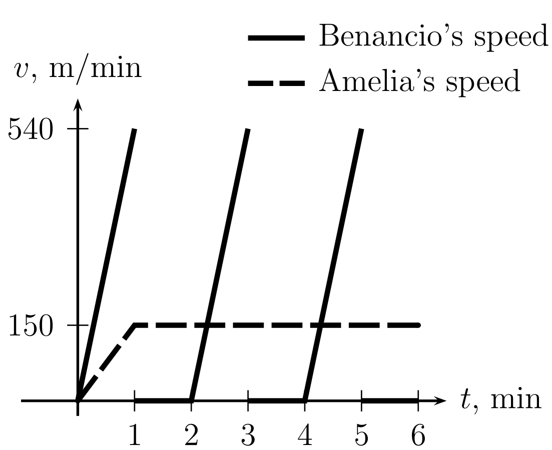
   (a) Suppose the base of the ladder is at the point $(u, 0)$. Where on the y-axis is the top of the ladder? Draw a picture!

   (b) Suppose you are standing at $(x, 0)$ and looking north (up the page). If $x < u$, how far away do you see the ladder?

   To be continued...

5. (Adapted from a Winter 2009 Math 115 Exam.) Amelia and Benancio, after much friendly trash talk about who's the fastest runner, decide to have a race. The two employ very different approaches.

   - Amelia takes the first minute to accelerate to a slow and steady pace which she maintains through the remainder of the race.
   - Benancio, on the other hand, spends the first minute accelerating to faster and faster speeds until he's exhausted and has to stop and rest for a minute—and then he repeats this process until the race is over. The graph below shows their speeds (in meters per minute), t minutes into the race. (Assume that the pattern shown continues for the duration of the race.)



   (a) What is Amelia's average speed over the first two minutes of the race? What is Benancio's?

   (b) Benancio immediately gets ahead of Amelia at the start of the race. How many minutes into the race does Amelia catch up to Benancio for the first time?

   (c) Draw graphs of Benancio and Amelia's positions at time t. Be as precise as possible.

   (d) If the race is 1080 meters total, who wins? What if it's 1081 meters?