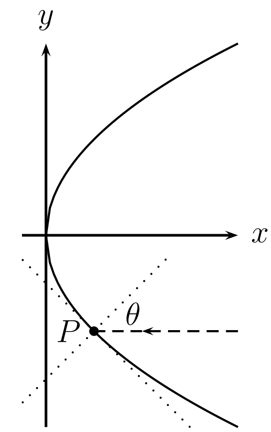
Douglass Houghton Workshop, Section 2, Thu 11/3/11

Worksheet Labradoodle

1. Last time we thought about a parabolic mirror in the shape of the graph of $y = \pm\sqrt{4x}$. So far we've found:



   - A light ray $y = -b$ hits the mirror at $P = (b^2/4, -b)$.
   - The slope of the tangent at that point is $-2/b$.
   - The normal line at the same point has slope $b/2$.
   - A line that makes an angle θ with the x-axis has slope $\tan\theta$.
   - So if we call the angle between the normal line and the horizontal θ, then $\tan\theta = b/2$.
   - If a light ray bounces off a mirror, the angle between the incoming ray and the normal line is the same as the angle between the outgoing ray with the normal line.

   (a) To the ray, the mirror looks flat, just like the tangent line. Draw the reflected ray. What angle does it make with the x-axis?

   (b) What is the slope of the reflected ray? Put your answer in terms of b. Hint: $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$.

   (c) Write an equation for the reflected ray.

   (d) Where does the reflected ray intersect the x-axis? What is surprising about this answer?

   (e) Graph several rays, with their reflections.

   (f) What's cool about this type of mirror?

2. In "The 12 days of Christmas", a certain poultry-afficianado receives a number of gifts from her true love:

   Day 1: A partridge in a pear tree. How to get it down?

   Day 2: 2 turtle doves, and another partridge in a pear tree. Is it the same tree?

   Day 3: 3 French hens, 2 more turtle doves, and another partidge.

   ...

   Day 12: 12 drummers drumming (loudly), eleven pipers piping (make them stop!), ..., and yet another partridge in a pear tree.

   (a) If item 1 is "partridge", item 2 is "turtle dove", etc., then write a formula for the total number of item n's received.

   (b) Of which item does Mr. Truelove send the most? (Solve using calculus.)

3. The three cities in the pictures below are at the corners of a 45°-45°-90° triangle whose legs are 10 miles long. The three mayors, working together, would like to build roads between them in such a way that there is a way to get from any one city to any other city.

(Say, A is Ann Arbor, B is Flint, and C is Port Huron.) The first, simple proposal (on the left) is to build a road from A to B and another from B to C. That would certainly work. But roads are expensive, and one of the mayors (who, luckily, studied calculus) proposes building roads from A and C to a point D just south of B, then building a road north from there to B.

(a) Let x be the length of the north-south road in the second proposal. What does it mean if $x = 0$?

(b) Calculate the total length of the new network in terms of x. Hint: "Law of cosines".

(c) Can you find a value of x which will produce a shorter network than the simple proposal?

4. (This problem appeared on a Fall, 2008 Math 115 exam) Determine a and b for the function of the form $y = f(t) = at^2 + b/t$, with a local minimum at $(1, 12)$.

5. Section 3.8 of your book (which we skip in 115) is about the "hyperbolic trig functions":

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

They are often called the even and odd parts of e^x, because they sum to e^x and one is an even function and one is an odd function.

(a) Which is which?

(b) Let $f(x)$ be any old function which is defined for all real numbers x. Think of a way to split $f(x)$ into even and odd parts. (Hint: Stare at the definitions above until you get an idea. Then check it.)

(c) cosh and sinh obey many rules similar, but not exactly the same, as those for cos and sin. To deduce a few, find the derivatives of $\cosh(x)$ and $\sinh(x)$. Then find $\cosh(2x)$ and $\sinh(2x)$. Can you find something resembling $\sin^2 x + \cos^2 x = 1$?

6. (This problem appeared on a Fall, 2007 Math 115 exam) Find the equations of all lines through the origin that are tangent to the parabola

$$y = x^2 - x + 4.$$