Douglass Houghton Workshop, Section 2, Tue 10/25/11

Worksheet Journey into Night

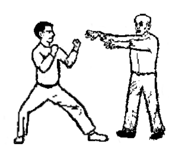
1. Suppose you construct a $1/z$ scale model of the White House, in order to film it blowing up. You will show the film at 24 frames per second. How many frames per second should you *film* so that when you slow the speed down, things will fall at believable speeds?

   Hints:

   - A falling object has a constant acceleration. What does that mean about its velocity at time t?
   - What does *that* mean about its height at time t?
   - So how long would it take an object to fall off the top of the White House (say, 50 ft high)?

2. There are at least three types of people: Humans, Zombies, and Dead Zombies. If there are some zombies and some humans, then when they meet some zombies will bite humans and turn them into zombies, and some humans will kill zombies and turn them into dead zombies. Also, sometimes zombies just die from lack of brains. Our job is to model the epidemic. Let

$$\begin{aligned} H(t) &= \text{The number of humans at the end of day } t \\ Z(t) &= \text{The number of zombies at the end of day } t \\ D(t) &= \text{The number of dead zombies at the end of day } t \end{aligned}$$



   (a) What are the changes in the number of humans from day t to day $t+1$? Write it like this:

$$H(t+1) = H(t) + \boxed{}$$

   The change can depend on $H(t)$, $Z(t)$, and $D(t)$.

   (b) Likewise:

$$Z(t+1) = Z(t) + \boxed{}$$

$$D(t+1) = D(t) + \boxed{}$$

3. Let $f(x) = x^2 - 2x + 4$ and $g(x) = -x^2 - 2x - 3$.

   (a) Draw $y = f(x)$ and $y = g(x)$ on the same set of axes. How many lines are tangent to both graphs?

   (b) Find the equations of those lines.

4. Consider a mirror in the shape of the graph of $y = \pm\sqrt{4x}$.

   (a) Draw the mirror (make it big). What shape is it?

   (b) Draw a light ray travelling leftward along the line $y = -b$, where b is some positive number (making $-b$ negative). At what point P does the ray hit the mirror?

   (c) Find, in terms of b, the slope of the tangent to the mirror at P.

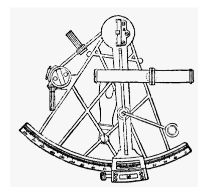
   (d) The *normal* to a curve at a point is the line through that point which is perpendicular to the tangent line. Find the slope of the normal to the mirror at P, and draw both the normal and tangent lines on your graph.

   (e) Suppose a line makes an angle θ with the x-axis. What is the slope of the line?

   (f) Let θ be the angle the normal to the mirror at P makes with the light ray $y = -b$. Can you write θ in terms of b? Hint: Use (4d) and (4e).

   To be continued...

5. Grace is painting a still-life of Burton Tower, from a distance of 100 ft away. She wonders how high it is. She pulls out her sextant (picture on the the right) and measures the angle to the top of the tower. The sextant has some inherent uncertainty; so let's say that Grace measures an angle θ, with an uncertainty of ϵ. So she knows that the real angle is somewhere between $\theta - \epsilon$ and $\theta + \epsilon$.



   (a) If the angle is exactly θ, how high is the tower?

   (b) Use a derivative to approximate how much uncertainty in that answer results from using the sextant. (Assume ϵ is small.)

   (c) Next she tries dropping a tennis ball (because she plays tennis) from the top of the tower, and measuring how long it takes to fall. She uses the work from problem 1 to estimate the height. If her stopwatch is accurate to within a time δ, about how accurate is her estimate of the height?

   (d) Suppose that $\epsilon = 0.01$ radians, $\delta = 0.1$ sec, and two measurements Grace makes are $\theta = 1.13$ radians and falling time of 3.6 seconds. What are the two estimates for the height of the tower, and which is more accurate?

6. Let's see if we can prove that the derivative of $\sin(x)$ is $\cos(x)$. Remember that last time we showed that

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

   (a) Show that $\lim_{\theta \to 0} \frac{1-\cos(\theta)}{\theta} = 0$. Hint: Multiply the top and bottom by $1 + \cos(\theta)$, and simplify.

   (b) Write down the definition of the derivative of $\sin(x)$ at $x = a$.

   (c) Use a trig identity to write $\sin(a + h)$ in terms of sines and cosines of a and h.

   (d) Now use the two limits we know (the one from last time and the one in part (a)) to simplify the derivative.