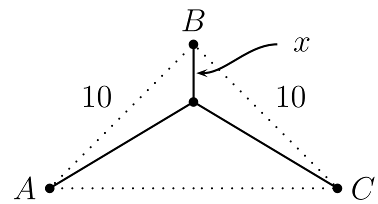
Douglass Houghton Workshop, Section 1, Mon 11/7/11

Worksheet May the Road Rise to Meet You

1. We've been working on the problem of finding the shortest road network between three cities in the plane.

   In the case we considered, the three cities were at the corners of a 45°-45°-90° triangle with legs 10 miles long. The simplest idea is to just build roads along the legs; that makes a network of length 20. By constructing a Y-shaped network like the one at the right, we found

   - The length of the Y network is $x + 2\sqrt{100 - 20x\cos(45) + x^2}$.

   (a) When $x = 0$, we have the V-shaped network. What's the length in that case?

   (b) Is there a value for x which makes the Y shorter than the V?

   (c) What if x is negative? Could we do better than the V that way?

   (d) Find the best value for x.

2. Let a be some positive constant. Graph $f(x) = e^{-\frac{1}{2}(ax)^2}$ on the board.

   (a) Does the graph look familiar?

   (b) Compute $f''(x)$.

   (c) What is an inflection point?

   (d) Find the inflection points of f.

   (e) Describe the effect of changing a.

3. In "The 12 days of Christmas", a certain poultry-afficianado receives a number of gifts from her true love:

   Day 1: A partridge in a pear tree. How to get it down?

   Day 2: 2 turtle doves, and another partridge in a pear tree. Is it the same tree?

   Day 3: 3 French hens, 2 more turtle doves, and another partidge.

   . . .

   Day 12: 12 drummers drumming (loudly), eleven pipers piping (make them stop!), . . ., and yet another partridge in a pear tree.

   (a) If item 1 is "partridge", item 2 is "turtle dove", etc., then write a formula for the total number of item n's received.

   (b) Of which item does Mr. Truelove send the most? (Solve using calculus.)

4. (This problem appeared on a Winter, 2008 Math 115 Exam.)

   (a) Consider the function $f(x) = x\sqrt{x+1}$. What is the domain of f?

   (b) Find all critical points, local maxima, and local minima of f.

   (c) Which of the local maxima and minima are global maxima / minima?

5. (This problem appeared on a Winter, 2009 Math 115 Exam) Suppose a is a positive (non-zero) constant, and consider the function

$$f(x) = \frac{1}{3}x^3 - 4a^2x.$$

   Determine all maxima and minima of f in the interval $[-3a, 5a]$. For each, specify whether it is global or local.

6. (This problem appeared on a Winter, 2005 Math 115 Exam) A family of functions is given by $r(x) = \frac{a}{x}e^{bx}$ for a, b, and $x > 0$.

   (a) For what values of a and b does the graph of r have a local minimum at the point $(4, 5)$? Show your work and all supporting evidence that your function satisfies the given properties.

   (b) Write an explicit formula for $r(x)$.

   (c) Is the graph of r concave up or down for $x > 0$? Explain using arguments based on calculus—not only from a graph.

7. (This problem appeared on a Fall, 2007 Math 115 exam) Find the equations of all lines through the origin that are tangent to the parabola

$$y = x^2 - x + 3.$$

8. (This problem appeared on a Winter, 2004 Math 115 exam. Really!) While exploring an exotic spring break location, you discover a colony of geese who lay golden eggs. You bring 20 geese back with you. Suppose each goose can lay 294 golden eggs per year. You decide maybe 20 geese isn't enough, so you consider getting some more of these magical creatures. However, for each extra goose you bring home there are less resources for all the geese. Therefore, for each new goose the amount of eggs produced will decrease by 7 eggs per goose per year. How many more geese should you bring back if you want to maximize the number of golden eggs per year laid?