Math 116 — Practice for Exam 1

Generated February 4, 2018

Name: SOLUTIONS

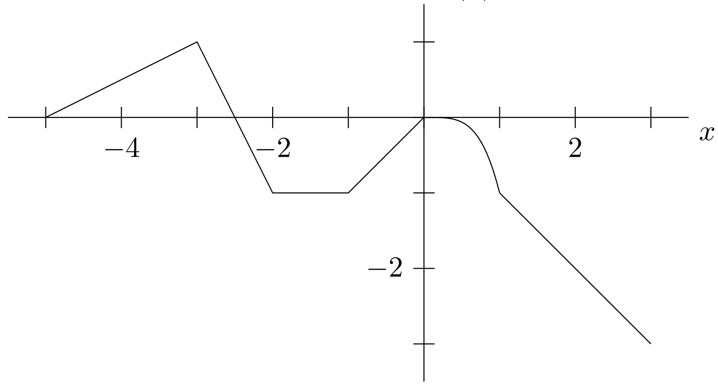
Instructor: ______________________ Section Number: __________

1. This exam has 17 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2015	1	5		11	
Winter 2016	1	1	catapult	16	
Fall 2017	1	5	solar panel	10	
Fall 2016	1	2	sparrows	16	
Winter 2017	1	2	plankton	13	
Winter 2013	1	8	tortoise+hare	11	
Winter 2015	1	7		16	
Fall 2016	1	7		6	
Winter 2009	1	1	containers	85	
Winter 2017	1	7		10	
Fall 2015	1	10	thumbtack doorknob	8	
Fall 2016	1	3	fishtank	11	
Fall 2013	1	4	shawarma kafta	17	
Winter 2010	1	3	spaceship	8	
Fall 2013	3	9	olive oil	13	
Winter 2017	1	9	nano pyramid	11	
Fall 2006	2	8	chain	16	
Total				278	

Recommended time (based on points): 208 minutes

5. [11 points] The following is the graph of a function $g(x)$.



Note that $g(x)$ is piecewise linear on [-5,0] and linear on [1,3].

a. [5 points] Estimate $\int_1^3 e^{-g(x)}dx$ using MID(2). Write out all the terms of your sum as well as your final answer.

Solution:
$$\text{MID}(2) = \frac{3-1}{2}\left(e^{-g(1.5)} + e^{-g(2.5)}\right) = e^{1.5} + e^{2.5} \approx 16.664$$

b. [3 points] If you were estimating $\int_0^1 \frac{dx}{g(x)+2}$ using LEFT(n), would your estimate be an overestimate, underestimate, or is there not enough information to tell? Circle your answer. You do not need to show your work or explain your answer.

Overestimate $\boxed{\text{Underestimate}}$ Not Enough Information

Solution: $\frac{d}{dx}\left(\frac{1}{g(x)+2}\right) = \frac{-g'(x)}{(g(x)+2)^2} > 0$, so LEFT($n$) is an underestimate.

c. [3 points] If you were estimating $\int_0^1 \frac{dx}{g(x)+2}$ using TRAP(n), would your estimate be an overestimate, underestimate, or is there not enough information to tell? Circle your answer. You do not need to show your work or explain your answer.

$\boxed{\text{Overestimate}}$ Underestimate Not Enough Information

Solution: $\frac{d^2}{dx^2}\left(\frac{1}{g(x)+2}\right) = \frac{-g''(x)(g(x)+2)^2 + 2(g'(x))^2(g(x)+2)}{(g(x)+2)^4} > 0$, so TRAP($n$) is an overestimate.

1. [16 points] At a time t seconds after a catapult throws a rock, the rock has horizontal velocity $v(t)$ m/s. Assume $v(t)$ is monotonic between the values given in the table and does not change concavity.

t	0	1	2	3	4	5	6	7	8
$v(t)$	47	34	24	16	10	6	3	1	0

a. [4 points] Estimate the average horizontal velocity of the rock between $t = 2$ and $t = 5$ using the trapezoid rule with 3 subdivisions. Write all the terms in your sum. Include **units**.

Solution:

$$\frac{\int_2^5 v(t)\,dt}{5-2} = \frac{Left(3) + Right(3)}{2 \cdot 3} = \frac{(v(2) + v(3) + v(4)) + (v(3) + v(4) + v(5))}{6} =$$

$$= \frac{24 + 16 + 10 + 16 + 10 + 6}{6} = \frac{82}{6} = \frac{41}{3}$$

The average horizontal velocity of the rock is 41/3 m/s.

b. [4 points] Estimate the total horizontal distance the rock traveled using a left Riemann sum with 8 subdivisions. Write all the terms in your sum. Include **units**.

Solution:

$$\int_0^8 v(t)\,dt = Left(8) = v(0) + v(1) + v(2) + v(3) + v(4) + v(5) + v(6) + v(7) =$$

$$= 47 + 34 + 24 + 16 + 10 + 6 + 3 + 1 = 141$$

The total horizontal distance the rock traveled is approximately 141 meters.

c. [4 points] Estimate the total horizontal distance the rock traveled using the midpoint rule with 4 subdivisions. Write all the terms in your sum. Include **units**.

Solution:

$$\int_0^8 v(t)\,dt = Mid(4) = 2(v(1) + v(3) + v(5) + v(7)) =$$
$$= 2(34 + 16 + 6 + 1) = 114$$

The total horizontal distance the rock traveled is approximately 114 meters.

d. [4 points] A second rock thrown by the catapult traveled horizontally 125 meters. Determine whether the first rock or the second rock traveled farther, or if there is not enough information to decide. Circle your answer. Justify your answer.

Solution:

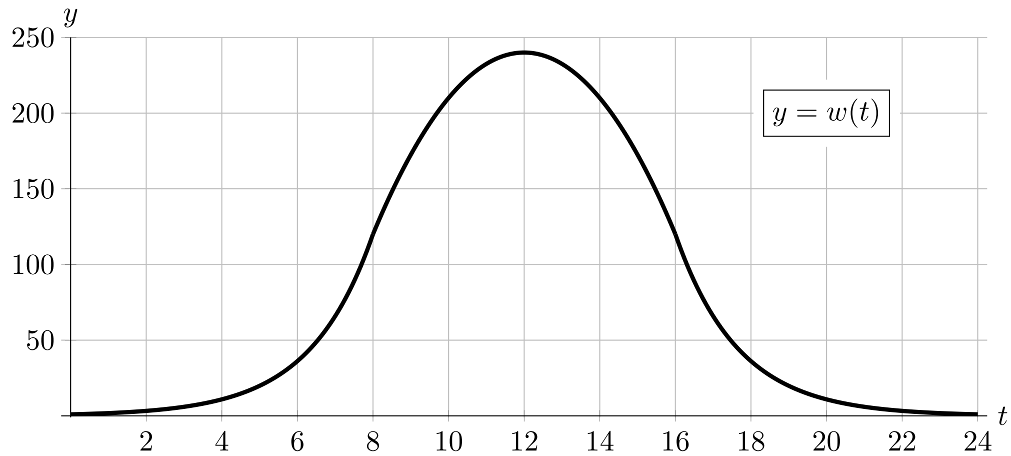
the first rock $\qquad$ $\boxed{\text{the second rock}}$ $\qquad$ not enough information

The function $v(t)$ is concave up since for example

$$-10 = \frac{v(2) - v(1)}{2 - 1} > \frac{v(1) - v(0)}{1 - 0} = -13$$

The trapezoid rule gives $Trap(4) = 121$ (or $Trap(8) = 117.5$). Since $v(t)$ is concave up, this is an overestimate which means that the first rock traveled at most 121 meters, that is less than the second.

5. [10 points] Suppose that the function $w(t)$ shown in the graph below models the power, in kilowatts, that is harvested at a particular solar panel installation in northern Norway at time t, where t is measured in hours after midnight on a typical summer day.



Consider the function W defined by

$$W(x) = \int_{2x}^{2x+4} w(t)\,dt.$$

Be sure to show your work very carefully on all parts of this problem.

a. [3 points] Estimate $W(4)$. In the context of this problem, what are the units on $W(4)$?

Solution: Note that W gives the area beneath the graph of w during a four-hour interval. In particular, $W(4) = \int_8^{12} w(t)\,dt$ is the area beneath the graph of $w(t)$ between the hours of $t = 8$ and $t = 12$. Estimating this integral (or estimating the area geometrically) gives $W(4) \approx 800$. The units on W are kilowatt·hours.

Answer: $W(4) \approx$ 800 **Units:** **kilowatt·hours**

b. [4 points] Estimate $W'(4)$. In the context of this problem, what are the units on $W'(4)$?

Solution: By the (first or second) Fundamental Theorem of Calculus together with the Chain Rule, we have

$$W'(x) = w(2x+4)\cdot(2) - w(2x)\cdot(2).$$

Substituting $x = 4$ gives

$$W'(4) = w(12)\cdot(2) - w(8)\cdot(2) = (240)\cdot(2) - (120)\cdot(2) = 240.$$

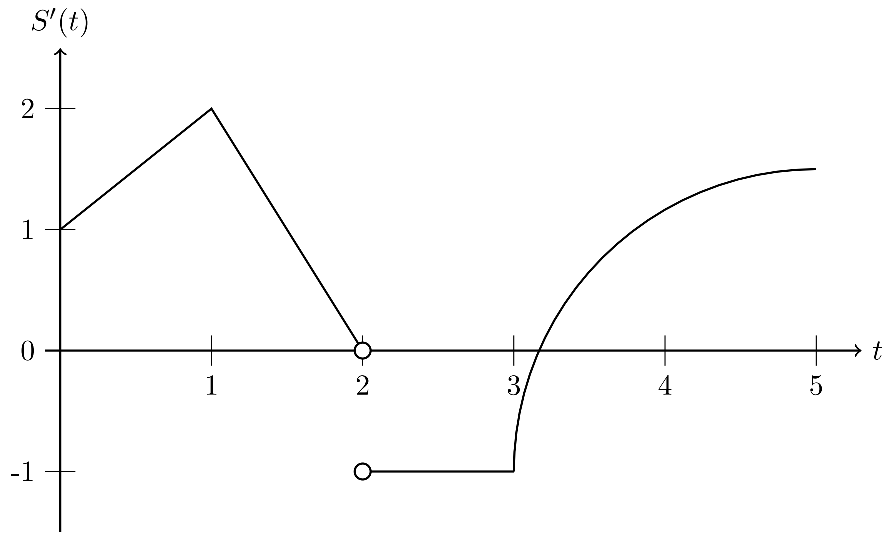
The units on W' are (kilowatts·hours)/(hours)=kilowatts.

Answer: $W'(4) \approx$ 240 **Units:** **kilowatts**

c. [3 points] Estimate the value(s) of x at which $W(x)$ attains its maximum value on the interval $0 \le x \le 8$. If there are no such values, explain why.

Solution: The function $W(x)$ gives the area beneath the graph of $w(t)$ during the four-hour interval between $t = 2x$ and $t = 2x + 4$. By inspecting the graph, one sees that this area is largest between the hours of $t = 10$ and $t = 14$, corresponding to $x = 5$. That is, $W(5)$ gives this maximal area, so $W(x)$ attains its maximum value at $x = 5$.

2. [16 points] The local sparrow population has been fluctuating unnaturally, and Raymond Green has five months of data to prove it. Let $S(t)$ denote the local sparrow population **in thousands**, t months after Green started collecting data. A graph of $S'(t)$, the rate of population growth, is below. Assume there are 2000 sparrows at $t = 1$.



a. [1 point] At which t-value(s) is the sparrow population increasing the fastest?

Solution: The population is increasing fastest at $t = 1$.

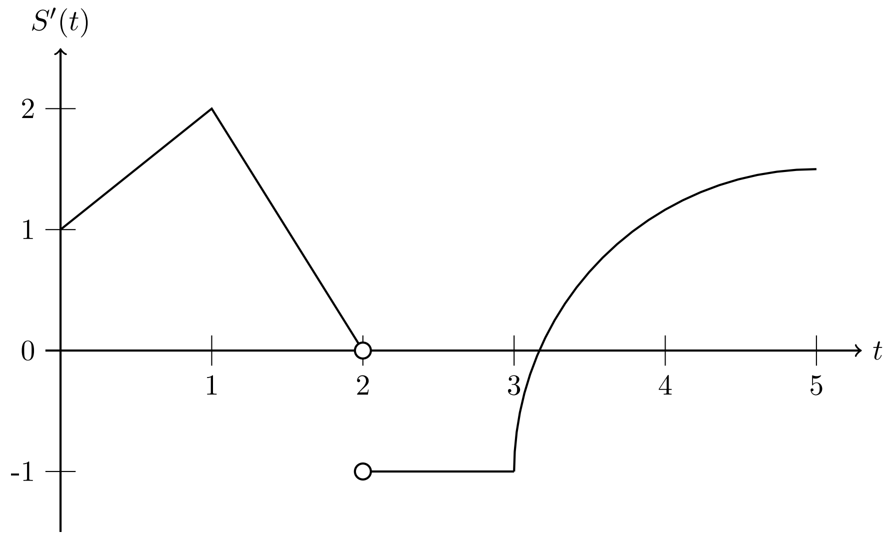
b. [3 points] What is the local sparrow population at $t = 0$, $t = 2$ and $t = 3$?

Solution: The population is 500 at $t = 0$, 3000 at $t = 2$, and 2000 at $t = 3$.

c. [2 points] At which t-values is the population at its highest and lowest?

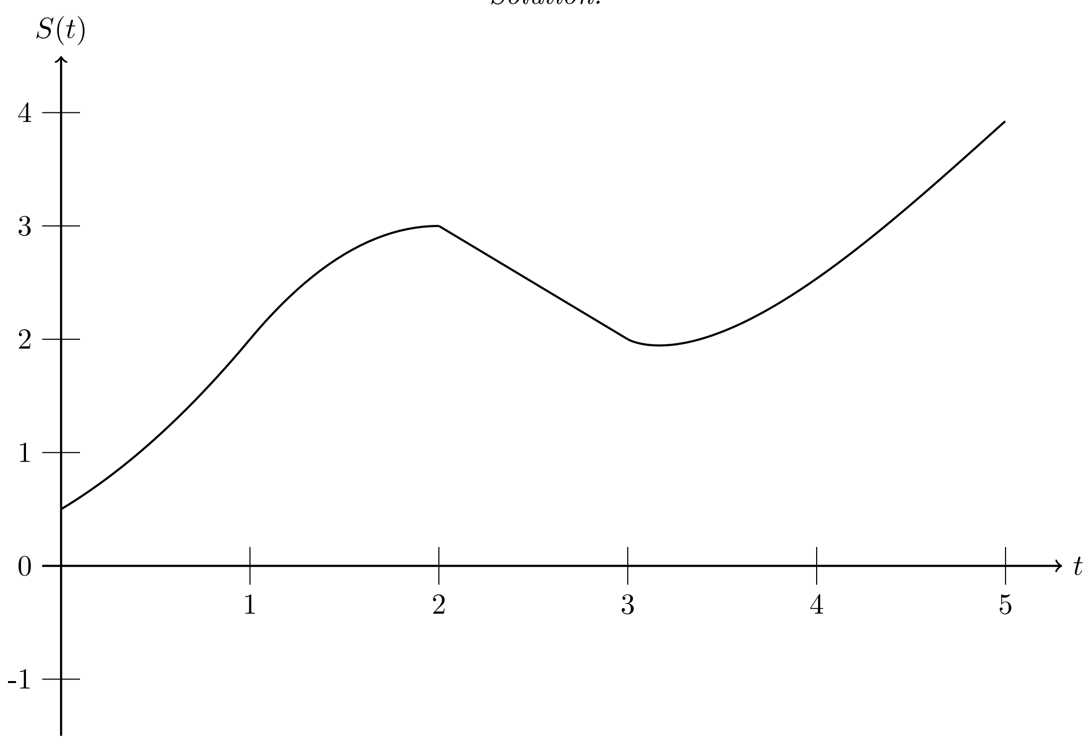
Solution: The population is highest at $t = 5$ and lowest at $t = 0$.

2 (continued). Recall that $S(t)$ is the local sparrow population **in thousands**, t months after Green began collecting data.

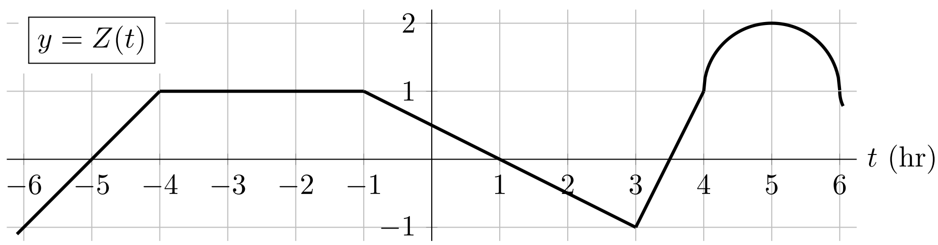


d. [10 points] Sketch a graph of $S(t)$ on the axes below, recalling that there are 2000 sparrows at $t = 1$. Label your vertical axis. Make sure that concavity and local extrema are clear.

Solution:



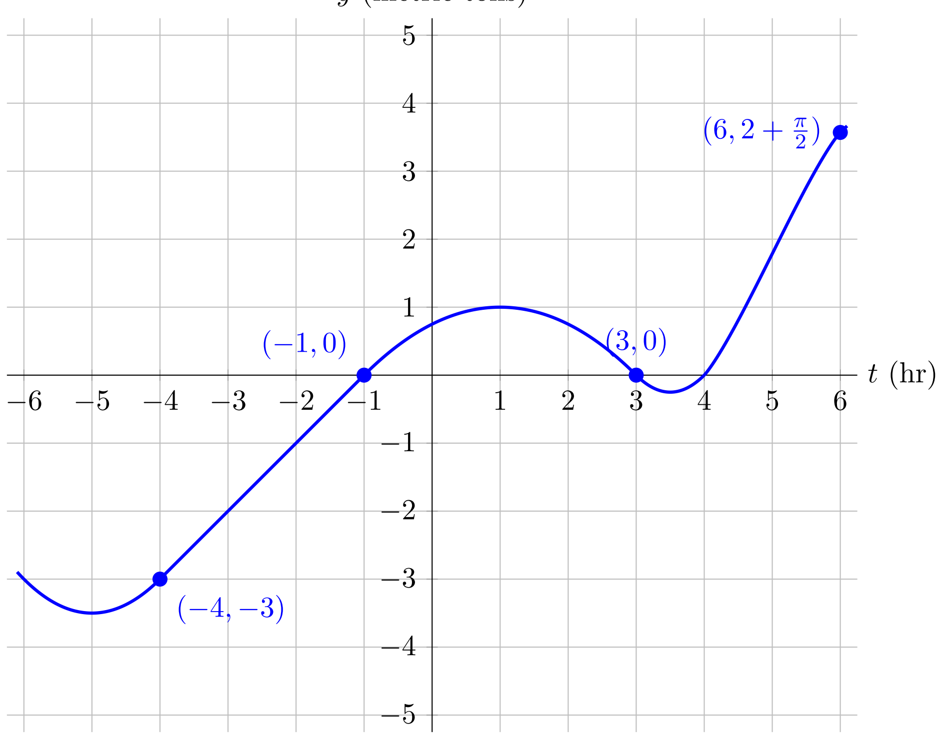
2. [13 points] Suppose $Z(t)$ is the rate of change, in metric tons per hour, of the biomass (i.e. total mass) of zooplankton in Loch Ness t hours after 8am on January 25, 2017.
Below is a portion of the graph of $Z(t)$. Note that this graph is linear on the intervals $[-6,-4]$, $[-4,-1]$, $[-1,3]$, and $[3,4]$. Also note that the portion of the graph for $4 \le t \le 6$ is the upper half of a circle centered at the point $(5,1)$.



Let $B(t)$ be the biomass, in metric tons, of zooplankton in Loch Ness t hours after 8am on January 25, 2017.

a. [10 points] Carefully sketch a graph of $y = B(t) - B(3)$ for $-6 \le t \le 6$ using the axes provided below. If there are features of this function that are difficult for you to draw, indicate these on your graph. Be sure that local extrema and concavity are clear.
Label the coordinates of the points on your graph at $t = -4, -1, 3, 6$.

Solution: Note that $B(t) - B(3) = \int_3^t Z(x)\,dx$ is the antiderivative of $Z(t)$ whose value is 0 at $t = 3$. Although it is difficult to tell here, the graph below is concave up for $4 < t < 5$ and concave down for $5 < t < 6$.



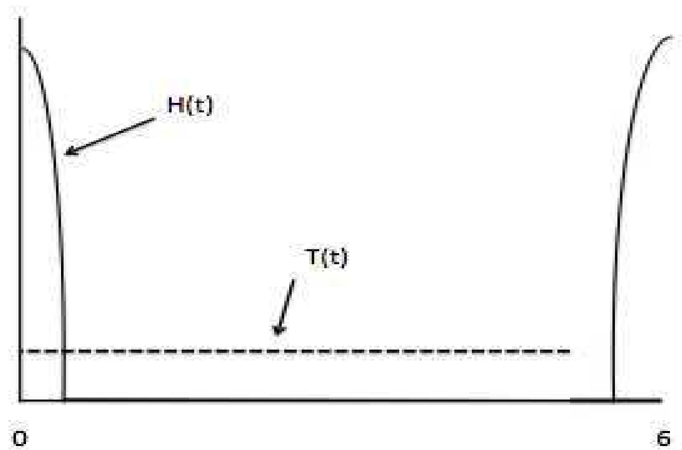
b. [3 points] Define $A(h)$ to be the average biomass (in metric tons) of zooplankton in Loch Ness during the first h hours after 8am on January 25, 2017. Write an expression for $A(h)$. (Your expression may involve integrals, the function Z, and/or the function B.)

Solution: $A(h)$ is the average value of the function $B(t)$ over the interval $0 \le t \le h$, so

$$A(h) = \frac{1}{h}\int_0^h B(t)\,dt.$$

8. [11 points] A tortoise and a hare decide to race. They decide to race a straight 5 kilometer course. The race starts at 12pm. The hare is much faster than the tortoise, so he's confident that he'll win. The hare runs very fast for 30 minutes, getting to what it knows is the half-way point. The hare is tired (it had been studying for exams the night before), so it decides to take a nap. It falls asleep for 5 hours, wakes up, discovers that (now that it's 5:30) it's dark, and runs to the finish line, arriving at 6pm. When it gets there, it's surprised to see the tortoise is already there. "I hope you enjoyed your nap! I've been here for an hour, since 5 o'clock!" the tortoise says. "Steady and slow is the way to go: I kept going the same speed the whole time."

Let $H(t)$ be the hare's velocity and $T(t)$ be the tortoise's velocity, in km per hour, where t is measured in hours after 12pm.



Let

$$R(t) = \int_0^t H(s)ds - \int_0^t T(s)ds.$$

a. [1 point] At times when $R(t) > 0$, who is winning the race?

Solution: The hare

b. [2 points] What is the practical interpretation of the function $|R(t)|$? Include units.

Solution: $|R(t)|$ is the distance in km between the tortoise and the hare t hours after 12pm.

c. [3 points] For what values of $0 \leq t \leq 6$, does $R(t) = 0$?

Solution: $t = 0, t = 2.5, t = 6$.

d. [2 points] For what values of $0 \leq t \leq 6$ is the function $\frac{dR}{dt} < 0$?

Solution: $0.5 < t < 5$.

e. [3 points] Write down a definite integral that represents the hare's average velocity from 12 to 12:30. What is the value of the hare's average velocity during this time?

Solution: $\frac{1}{.5}\int_0^{1/2} H(s)ds$. We know that $\int_0^{1/2} H(s)ds = 2.5$, because the Hare has gotten halfway by 12:30. Therefore, the average velocity is 5 km/hr.

7. [16 points] In each part, circle "True" if the statement is always true and circle "False" otherwise. No justification is necessary. Any unclear markings will be marked incorrect.

Solution:

a. [8 points] Suppose $g(x)$ is a positive function, defined for all real numbers x, with continuous first derivative.

(1) $\displaystyle\int_0^7 xg(x^2)\,dx = \int_0^7 g(u)\,du.$

True $\boxed{\text{False}}$

(2) $\displaystyle\int_0^7 xg(x^2)\,dx = \frac{1}{2}\int_0^{49} g(t)\,dt.$

$\boxed{\text{True}}$ False

(3) $\displaystyle\int_0^7 xg(x^2)\,dx = 7g(49) - \int_0^7 g(x^2)\,dx.$

True $\boxed{\text{False}}$

(4) $\displaystyle\int_0^7 xg(x^2)\,dx = \frac{49}{2}g(49) - \int_0^7 x^3g'(x^2)\,dx.$

$\boxed{\text{True}}$ False

b. [8 points] Suppose $h(y)$ is the density, in grams per cm, of a thin rod of length 10 cm, y cm from one end. Suppose the rod has mass M.

(1) $\displaystyle\int_0^5 h(y)\,dy = \frac{M}{2}.$

True $\boxed{\text{False}}$

(2) The center of mass of the rod is $\displaystyle\int_0^{10} yh(y)\,dy.$

True $\boxed{\text{False}}$

(3) If $h(y)$ is a constant function, then $h(y) = \frac{M}{10}$.

$\boxed{\text{True}}$ False

(4) The average value of $h(y)$ on $[0, 10]$ is $\frac{M}{10}$.

$\boxed{\text{True}}$ False

7. [6 points] Suppose that g is a continuous function, and define another function G by

$$G(x) = \int_0^x g(t)\,dt.$$

Given that $\int_0^7 g(x)\,dx = 5$, compute

$$\int_0^7 g(x)(G(x))^2\,dx.$$

Show each step of your computation.

Solution: Substitution gives

$$\int_0^7 g(x)(G(x))^2\,dx = \int_{G(0)}^{G(7)} u^2\,du = \left.\frac{u^3}{3}\right|_0^5 = \frac{125}{3}.$$

Alternatively, integrate by parts to obtain

$$\int_0^7 g(x)(G(x))^2\,dx = (G(x))^3\Big|_0^7 - 2\int_0^7 g(x)(G(x))^2\,dx,$$

which after rearranging gives

$$\int_0^7 g(x)(G(x))^2\,dx = \frac{1}{3}\left((G(x))^3\Big|_0^7\right) = \frac{125}{3}.$$

1. (85 points) **Modeling the amount of water in a container.** Consider 3 containers, in which water flows into or out of each container at a different rate. Your job is to determine how much water is in each container at the end of 75 seconds.

   a. If $r(t)$ describes the flow of water into a container with units of milliliters per second (ml/sec), and t is measured in seconds, write a sentence or two explaining what $\int_a^b r(t)dt$ means in this context.

   This definite integral represents the net change in the amount of water (in ml) in the container between $t = a$ seconds and $t = b$ seconds

   b. ***Container 1*:** The initial amount of water in container 1 is 150 milliliters (ml). Water flows into container 1 at a rate $r_1(t)$ ml/sec described by the following data.

Time (sec)	0	25	50	75
$r_1(t)$ (ml/sec)	23	21	6	2

   What is the volume of water at the end of 75 seconds? Describe the method you use, and the accuracy of your method (i.e. exact, over/underestimate, etc.). If you've made assumptions that affect your answer, you should also explain those as well.

   LHS: 150+23(25)+21(25)+6(25)=150+1250=1400 ml. This is an overestimate if we assume that the data is decreasing always.

   RHS: 150+21(25)+6(25)+2(25)=150+725=875 ml. This is an underestimate if we assume that the data is always increasing.

   TRAP=(RHS+LSH)/2=987.5. If students assume data is linear, then trap is exact. Otherwise, we need more info to know if this estimation is over/under.

c. ***Container 2:*** The initial amount of water in container 2 is 150 milliliters (ml). Water flows into container 2 at a rate $r_2(t)$ ml/sec. *An anti-derivative* of $r_2(t)$ is $R_2(t) = \frac{100t}{35}\sin\left(\frac{t}{18}+3\right)$. What is the volume of water at the end of 75 seconds? Describe the method you use, and the accuracy of your method (i.e. exact, over/underestimate, etc.). If you've made assumptions that affect your answer, you should also explain those as well.

150+ $R_2(75) - R_2(0) \approx$ 150+165.63=315.63. This is an exact method. But round-off error may occur, depending on how student writes down answer.

d. ***Container 3:*** The initial amount of water in container 3 is 150 milliliters (ml). Water flows into container 3 at a rate $r_3(t) = \frac{50}{t^2+5t+6} + 10\sin\left(\frac{2\pi}{75}t\right)$ ml/sec. What is the volume of water at the end of 75 seconds? Describe the method you use, and the accuracy of your method (i.e. exact, over/underestimate, etc.). If you've made assumptions that affect your answer, you should also explain those as well.

Find the anti-derivative of $r_3(t)$ and use FTC. This requires partial fractions and a straightforward w-substitution.

$$\int_0^{75} \frac{50}{t^2+5t+6} + 10\sin\left(\frac{2\pi}{75}t\right)dt = \int_0^{75} \frac{50}{t+2} - \frac{50}{t+3} + 10\sin\left(\frac{2\pi}{75}t\right)dt$$

$$= 50\ln|t+2| - 50\ln|t+3| - \frac{750}{2\pi}\cos\left(\frac{2\pi}{75}t\right)\Big|_0^{75}$$

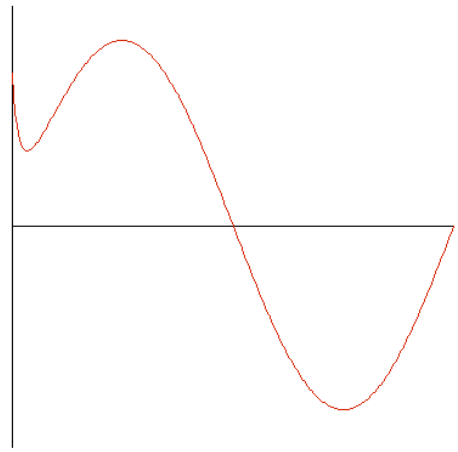
$$= \left(50\ln\left(\frac{77}{78}\right) - \frac{750}{2\pi}\cos(2\pi)\right) - \left(50\ln\left(\frac{2}{3}\right) - \frac{750}{2\pi}\cos(2\pi)\right)$$

$$= 50\ln\left(\frac{77}{52}\right) \approx 19.62$$

Add this to 150 to get 169.62. This method is exact.

e. Considering only the first 75 seconds, does container 3 have its maximum amount of water at 75 seconds? Justify your response.

No, $r_3(t)$ becomes negative prior to 75 seconds. This means water is leaving the container when the rate function is negative. I can graph $r_3(t)$ and see that $r_3(t)$ is negative from about 38 seconds to 75 seconds. This means that the container is losing water and cannot have its maximum at 75 seconds.



7. [10 points] Consider the function F defined for all x by the formula

$$F(x) = \int_7^{x^2} e^{-t^2}\,dt.$$

a. [1 point] Find a number $a \geq 0$ so that $F(a) = 0$.

Solution: $a = \sqrt{7}$.

b. [4 points]

(i) Calculate $F'(x)$. Your answer should not contain any integrals.

Solution: Applying the Second Fundamental Theorem of Calculus and the Chain Rule, we find

$$F'(x) = e^{-(x^2)^2} \cdot 2x = 2xe^{-x^4}.$$

(ii) Is $F(x)$ increasing on the entire interval $[1, 8]$? Why or why not?

Solution: $F'(x) > 0$ if $x > 1$ (in fact, if $x > 0$). Thus $F(x)$ is increasing on this interval. Alternatively, $F(x)$ is the integral of a positive function, and the interval expands as x increases.

c. [3 points] Write out each term of a MID(3) estimate of $F(5)$.
(You do **not** need to find or approximate the numerical value of your answer.)

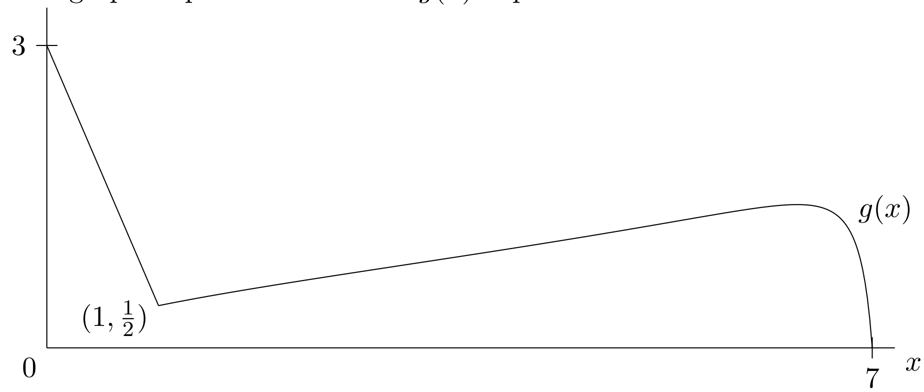
Solution: $F(5) = \displaystyle\int_7^{25} e^{-x^2}\,dx$, and therefore

$$MID(3) = 6 \cdot (e^{-10^2} + e^{-16^2} + e^{-22^2}) = 6(e^{-100} + e^{-256} + e^{-484}).$$

d. [2 points] Is your answer to part (c) an overestimate or underestimate of $F(5)$?
Briefly explain your reasoning.

Solution: Since e^{-x^2} is concave up on the interval $[7, 25]$, MID(3) is an underestimate of $F(5)$.

10. [8 points] The graph of part of a function $g(x)$ is pictured below.



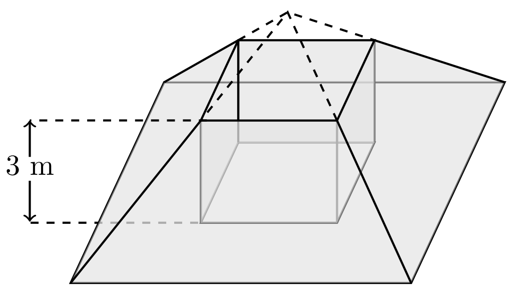
a. [4 points] A thumbtack has the shape of the solid obtained by rotating the region bounded by $y = g(x)$, the x-axis and y-axis, about the y-axis. Find an expression involving integrals that gives the volume of the thumbtack. Do not evaluate any integrals.

Solution: Using the cylindrical shell method, the volume of the thumbtack is $\int_0^7 2\pi x g(x)$.

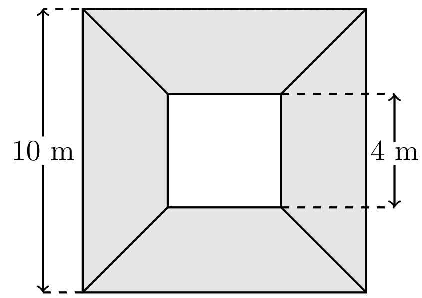
b. [4 points] A door knob has the shape of the solid obtained by rotating the region bounded by $y = g(x)$, the x-axis and y-axis, about the x-axis. Find an expression involving integrals that gives the volume of the door knob. Do not evaluate any integrals.

Solution: Using the washer method, the volume of the door knob is $\int_0^7 \pi(g(x))^2 dx$.

3. [11 points] During a trip to the local aquarium, Steph becomes curious and decides to taste the fish food. The fish food tank is completely filled with food, and it is in the shape of a pyramid with a vertical hole through its center, illustrated below (the dashed lines are not part of the tank). The tank itself is 3 m tall, and the pyramid base is a square of side length 10 m. The top and bottom of the hole are squares of side length 4 m. The food is contained in the shaded region only, **not** in the hole.



SIDE VIEW



TOP VIEW

a. [5 points] Write an expression that gives the approximate volume of a slice of fish food of thickness Δh meters, h meters from the bottom of the tank.

Solution: The approximate volume is

$$((10-2h)^2 - 4^2)\Delta h \qquad \text{m}^3.$$

b. [3 points] Suppose that the mass density of fish food is a constant δ kg/m^3. Write, but do **not** evaluate, an expression involving integrals that gives the mass of fish food in the tank.

Solution: The mass of fish food in the tank is given by

$$\delta \int_0^3 ((10-2h)^2 - 4^2)\, dh \qquad \text{kg}.$$

c. [3 points] Write an expression involving integrals that gives $\overline{h}$, the h-coordinate of the center of mass of the fish food, where h is defined as above. Do **not** evaluate your expression.

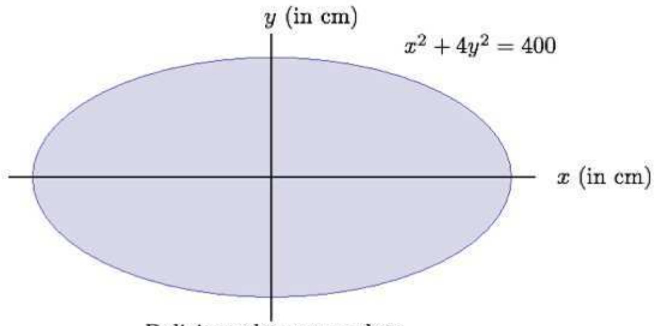
Solution: We have

$$\overline{h} = \frac{\int_0^3 h((10-2h)^2 - 4^2)\, dh}{\int_0^3 ((10-2h)^2 - 4^2)\, dh} \qquad \text{m}.$$

4. [17 points]

a. [8 points] The delicious chicken shawarma platter is served on an elliptical plate, described by the equation $x^2 + 4y^2 = 400$. The mass density of the platter, including the food, is a function of y, given by $\delta(y) = 10 + 0.5y$ grams per cm^2.

In this problem, you do not need to evaluate any integrals.



Delicious shawarma plate

i) (4 points) Find an expression containing a definite integral that computes the mass of the chicken shawarma platter (including the food).

Solution:

$$m = \int_{-10}^{10} 2\sqrt{400 - 4y^2} \cdot (10 + 0.5y)dy.$$

ii) (4 points) Find expressions for the coordinates $\bar{x}$, $\bar{y}$ of the center of mass of the platter. If your expression does not involve an integral, include a justification.

Solution: $\bar{x} = 0$, since both the shape and density function are symmetric about the y-axis.

$$\bar{y} = \frac{\int_{-10}^{10} y \cdot 2\sqrt{400 - 4y^2} \cdot (10 + 0.5y)dy}{\int_{-10}^{10} 2\sqrt{400 - 4y^2} \cdot (10 + 0.5y)dy}.$$

b. [9 points] The mouthwatering kafta kabob platter is served on a circular plate, with radius 20 cm. Including the food, the overall mass density of the platter is given by $\delta(r) = \dfrac{50}{2+r^2}$ grams per cm^2, where r is the distance from the center of the plate (in cm).

i) (4 points) Write a definite integral that computes the mass of the kafta kabob platter (including food). You do not need to evaluate the integral.

Solution:

$$m = \int_0^{20} 2\pi r \cdot \frac{50}{2+r^2} dr.$$

ii) (3 points) Write an estimate for your expression in part i) of the mass of the platter using LEFT(3). Show all the terms in the sum. You do not need to evaluate the sum.

Solution:

$$\begin{aligned}
\text{LEFT}(3) &= \frac{20}{3}\left(2\pi \cdot 0 \cdot \frac{50}{2+0^2} + 2\pi \cdot \frac{20}{3} \cdot \frac{50}{2+(\frac{20}{3})^2} + 2\pi \cdot \frac{40}{3} \cdot \frac{50}{2+(\frac{40}{3})^2} \right) \\
&= \frac{20}{3}\left(0 + 45.09 + 23.30 \right). \\
&= \quad 0 + 300.6 + 155.33.
\end{aligned}$$

iii) (2 points) Where is the center of mass of this platter? Justify.

Solution: At the center of the plate, since both the shape and density function are symmetric about the origin.

3. [8 points] When a spaceship takes off it does not travel in a straight path as it ascends. Instead, it turns slightly east, so that it gains speed by traveling with the rotation of the earth. From mission control's point of view, the spaceship's path appears to follow the curve $y = \sqrt{1+10x^2} - 1$, where y is the height in meters of the spaceship off the ground and x is the horizontal movement in meters from the launch pad. After 20 seconds, the spaceship appears to be 1 kilometer high.

a. [2 points] Determine, to the nearest hundredth of a meter, the horizontal distance the spaceship has traveled from the launch pad at 20 seconds.

Solution: When $y = 1000$ we solve for x:

$$\begin{aligned} 1000 &= \sqrt{1+10x^2} - 1 \\ 1001^2 &= 1 + 10x^2 \\ 1002000 &= 10x^2 \\ x &= \sqrt{100200} \approx 316.54 \text{ meters} \end{aligned}$$

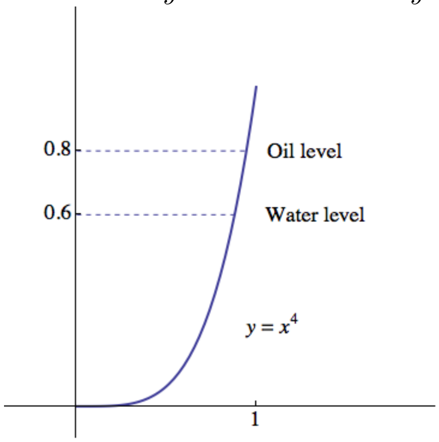
b. [6 points] From mission control's point of view, what is the total distance of the path the spaceship appears to have traveled throughout the first 20 seconds of its trip? Give your answer to the nearest hundredth of a meter, and be sure to show enough work to justify your answer.

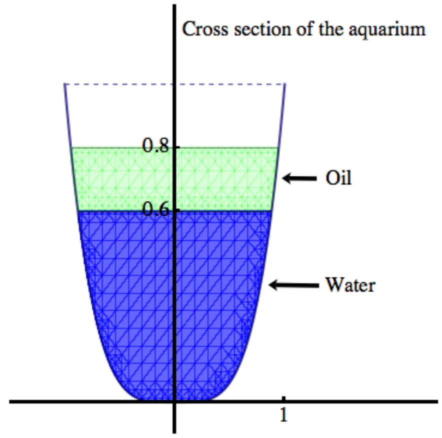
Solution: When $y = 0$ we can also see that $x = 0$. Now we use the formula to calculate arc length, noting that $\frac{dy}{dx} = \frac{10x}{\sqrt{1+10x^2}}$, and we get

$$\int_0^{316.543} \sqrt{1 + \frac{100x^2}{1+10x^2}}\,dx \approx 1048.95 \text{ meters},$$

by using the calculator to determine this integral.

9. [13 points] Olive oil have been poured into the Math Department's starfish aquarium! The shape of the aquarium is a solid of revolution, obtained by rotating the graph of $y = x^4$ for $0 \le x \le 1$ around the y-axis. Here x and y are measured in meters.





The aquarium contains water up to a level of $y = 0.6$ meters. There is a layer of oil of thickness 0.2 meters floating on top of the water. The water and olive oil have densities 1000 and 800 kg per m^3, respectively. Use the value of $g = 9.8$ m per s^2 for the acceleration due to gravity.

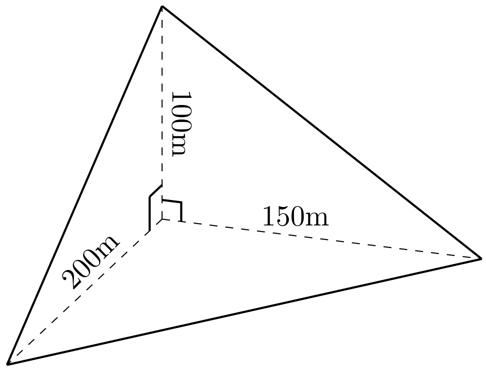
a. [6 points] Give an expression involving definite integrals that computes the total mass of the water in the aquarium.

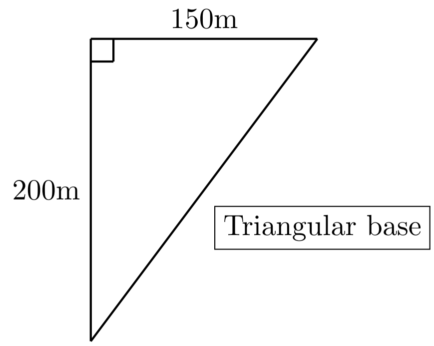
Solution: $\text{Mass}_{water} = \int_0^{0.6} \pi(\sqrt[4]{y})^2(1000)dy = \int_0^{0.6} \pi\sqrt{y}(1000)dy$

b. [7 points] Give an expression involving definite integrals that computes the work necessary to pump all the olive oil to the top of the aquarium.

Solution: $\text{Work}_{oil} = \int_{0.6}^{0.8} \pi(\sqrt[4]{y})^2(800)(9.8)(1-y)dy = \int_{0.6}^{0.8} \pi\sqrt{y}(800)(9.8)(1-y)dy$

9. [11 points] Advanced beings from another planet recently realized they left a stockpile of nanotechnology here on Earth. These tiny devices are stacked in the shape of a pyramid with a triangular base that is flat on the ground. Its base is a right triangle with perpendicular sides of length 150m and 200m. Two of the other three sides are also right triangles, and all three right angles meet at one corner at the base of the pile. The fourth side is a triangle whose sides are the hypotenuses of the other three triangles. (See diagrams below.)





The density of the contents of the pile at a height of h meters above the ground is given by

$$\delta(h) = \frac{2}{\sqrt{1+h^2}} \text{ kg/m}^3.$$

For this problem, you may assume the acceleration due to gravity is $g = 9.8$ m/s^2.

a. [4 points] Write an expression in terms of h that approximates the volume (in cubic meters) of a horizontal slice of thickness Δh of the contents of the pile at a distance h meters above the ground.

Solution: The cross section of the pile h meters above the ground is a right triangle with legs of length $200-2h$ and $150-1.5h$ meters. (One can use e.g. linearity or similar triangles to find these lengths.) Therefore, the volume (in cubic meters) of the horizontal slice is approximately $\frac{1}{2}(200-2h)(150-1.5h)\Delta h$.

b. [3 points] Write, but do **not** to evaluate, an integral that gives the total mass of the pile of nanotechnology. Include units.

Solution: We mutiply the expression in (a) by the density to get an approximation for the mass of the slice and integrate that expression from 0 to 100 while replacing Δh by dh to find a total mass of

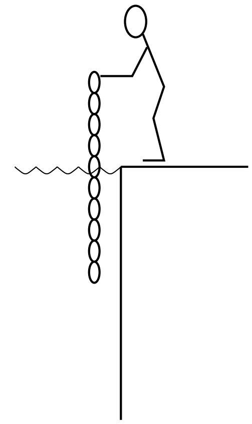
$$\int_0^{100} \frac{1}{2}(200-2h)(150-1.5h)\frac{2}{\sqrt{1+h^2}}\, dh \quad \text{kilograms.}$$

c. [4 points] The beings must return to Earth and collect the nanotech that they left behind. Suppose that the spaceship hovers 150 meters above the ground (directly above the pile) while recovering the nanotechnology. Write, but do **not** evaluate, an integral which gives the total work that must be done in order to lift all of the nanotech from the pile into the ship. Include units.

Solution: We multiply the expression in (a) by the density, acceleration due to gravity, and the distance required to lift it to a height of 150 m to estimate the work done to lift that slice up to the spaceship. Then we integrate this expression from 0 to 100 and replace Δh by dh to find that the total work done is

$$\int_0^{100} g\cdot(150-h)\cdot\frac{1}{2}(200-2h)(150-1.5h)\frac{2}{\sqrt{1+h^2}}\, dh \quad \text{Joules.}$$

8. [16 points] Chris is standing at the edge of a swimming pool, holding a chain that is partially submerged in the water of the pool, as shown in the figure to the right. The chain is six feet long and weighs 5 lb/ft. When it is in the water, however, the buoyant force of the water makes the effective weight of the chain less—in the water, it weighs only 3 lb/ft. If the chain is initially half submerged in the pool and Chris lifts it straight up until it is entirely out of the water, how much work does Chris do?



Solution:
We can find the total work by considering the work to move from a given position, x (measured as the distance that the chain has been raised), to the position $x + \Delta x$. The required force is the weight of the chain,

$$F = (\text{weight of chain above the water}) + (\text{weight of chain in the water}).$$

The length of chain above the water is $3 + x$ ft, and the length below is $3 - x$ ft. Thus $F = (3 + x)(5) + (3 - x)(3) = 24 + 2x$ lb. The work to lift the chain through this distance Δx is then $\Delta W = (24 + 2x)\Delta x$. The total work is found by integrating over the 3 ft that it is lifted, so

$$W = \int_0^3 (24 + 2x)\,dx = (24x + x^2)\Big|_0^3 = 72 + 9 = 81\,\text{ft}\cdot\text{lb}.$$

An alternate solution is to consider the top half and bottom half of the chain separately. The top half all moves 3 ft and has a constant weight, so the work is $W_t = ((3\,\text{ft})(5\,\text{lb/ft}))(3\,\text{ft}) = 45\,\text{ft}\cdot\text{lb}$. Then, a piece of the bottom part of chain that has length Δx and is x feet from the bottom of the chain's initial position has a weight 3 lb/ft for the distance $3 - x$ and a weight 5 lb/ft for the distance x. Thus the work to lift it is $\Delta W = (3\Delta x)(3 - x) + (5\Delta x)x = (9 + 2x)\Delta x$. The total work to lift the bottom half of the chain is then $W_b = \int_0^3 9 + 2x\,dx = 9x + x^2\big|_0^3 = 27 + 9 = 36\,\text{ft}\cdot\text{lb}$. The total work is the sum of W_t and W_b, which is not surprisingly still 81 ft·lb.