Math 116 — Practice for Exam 3

Generated April 15, 2012

Name: ______________________________

Instructor: ______________________ Section Number: __________

1. This exam has 13 pages including this cover. There are 10 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2008	3	6	ibuprofin	12	
Fall 2011	3	7		12	
Fall 2008	3	7	snow	16	
Fall 2006	3	2		7	
Fall 2004	3	10	Doppler	8	
Fall 2011	2	8	spirals	10	
Fall 2010	3	3	sewage flow	11	
Fall 2006	2	1	corn	12	
Winter 2011	3	7	wine glass	12	
Fall 2009	1	8		13	
Total				113	

Recommended time (based on points): 125 minutes

6. [12 points] When a patient takes a drug (e.g., by ingesting a pill), the amount of the drug in her/his system changes with time. We can think of this process discretely (each pill is an immediately delivered dose) or continuously (each pill delivers a small amount of drug per unit time over a long time). This problem considers these two different models.

a. [4 points] Suppose that ibuprofen is taken in 200 mg doses every six hours, and that all 200 mg are delivered to the patient's body immediately when the pill is taken. After six hours, 12.5% of the ibuprofen remains. Find expressions for the amount of ibuprofen in the patient immediately before and after the nth pill taken. Include work; without work, you may receive no credit.

Solution: Immediately after the first pill, the amount in the patient's system is $A_1 = 200$. Immediately before the second pill is taken the amount in the patient's system is $B_2 = (0.125)(200)$, and immediately after, $A_2 = 200 + (0.125)(200)$. Similarly, we have $B_3 = (0.125)(200) + (0.125)^2(200)$ and $A_3 = 200 + (0.125)(200) + (0.125)^2(200)$, and so on. Thus immediately after the nth pill taken, the patient has

$$A_n = 200 + (0.125)(200) + \cdots + (0.125)^{n-1}(200) = \frac{200(1 - 0.125^n)}{1 - 0.125} \text{ mg},$$

or about $228.57(1 - 0.125^n)$ mg of ibuprofen in her/his system, and immediately before the nth pill the patient has (for $n > 1$),

$$B_n = (0.125)(200) + \cdots + (0.125)^{n-1}(200) = \frac{(0.125)(200)(1 - 0.125^{n-1})}{1 - 0.125} \text{ mg},$$

or about $28.571(1 - 0.125^{n-1})$ mg of ibuprofen in her/his system.

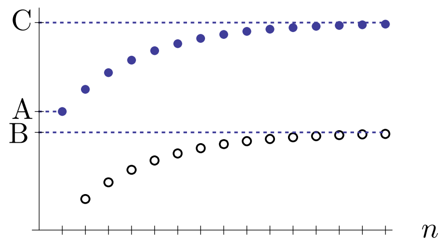
b. [4 points] Now suppose that ibuprofen is taken in a time-release capsule that continuously releases 35 mg/hr of ibuprofen per hour for six hours. The drug decays at a rate proportional to the amount in the body, with a constant of proportionality $r = 0.35$. Write a differential equation for the amount of ibuprofen, $y(t)$, in the patient as a function of time. Solve your differential equation, assuming that there is no ibuprofen in the patient initially.

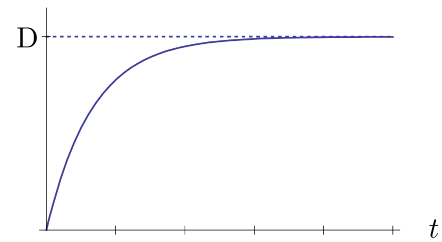
Solution: We know that the rate of change of the ibuprofen in the body, $y(t)$, is 35 mg/hr less the decay of the amount that's present, so we have $\frac{dy}{dt} = 35 - 0.35y$, and if the initial amount present is zero, we also have $y(0) = 0$. We can solve this by separation of variables:

$$\frac{dy}{dt} = 35 - 0.35y = -0.35(y - 100), \quad \text{so} \quad \frac{dy}{y - 100} = -0.35\,dt.$$

Integrating both sides, we have $\ln|y - 100| = -0.35t + C$, so that $y = 100 + k\,e^{-0.35t}$. Because $y(0) = 0$, $k = -100$, and we have $y = 100(1 - e^{-0.35t})$. Note that this is only valid for $0 \le t \le 6$; at $t = 6$ we expect that another pill will be taken, and we will then have to solve the same differential equation with the initial condition $y(6) = 100(1 - e^{-0.35(6)}) = 87.75$.

c. [4 points] *This is the third part of problem (6), which begins on the previous page.* The figure below shows graphs that may correspond to the solutions that you obtained in (6a) and (6b). The graph to the left corresponds to part (a), with the solid and open circles in the graph showing the amount of ibuprofen in the patient after and before the nth pill. The graph to the right corresponds to (b). Assuming that these graphs do represent your solutions, use your results from (a) and (b) to determine the values of A, B, C, and D.





Solution: We note that A is the amount of ibuprofen in the patient after taking one pill, which is 200 mg; C is the limiting value for this quantity, which is 228.6 mg. B is the limiting value for the amount of ibuprofen in the patient before the nth pill, which is 28.6 mg. D is the limit of the exponential in (b), which is 100 mg.

7. [16 points] The annual snowfall in Ann Arbor has mean 52.91 inches and standard deviation 12.67 inches. Assume that these data are normally distributed, and let x be the deviation of a year's snowfall from the mean (so that if $x = -2$ in a given year, it means that the snowfall that year was 50.91 inches). Then the probability density function for x is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/(2\sigma^2)} = 0.03149\, e^{-x^2/321.1},$$

so that its cumulative distribution function $P(x)$ is

$$P(x) = \int_{-\infty}^{x} p(t)\, dt = \int_{-\infty}^{x} 0.03149\, e^{-t^2/321.1}\, dt.$$

a. [2 points] Explain why $P(x) = \dfrac{1}{2} + \displaystyle\int_0^x 0.03149\, e^{-t^2/321.1}\, dt.$

Solution: We know the area under $p(x)$ is one, and because the normal distribution is symmetric about its mean, we know $\int_{-\infty}^{0} p(x)\, dx = \frac{1}{2}$. Thus $P(x) = \int_{-\infty}^{x} p(t)\, dt = \int_{-\infty}^{0} p(t)\, dt + \int_0^x p(t)\, dt = \frac{1}{2} + \int_0^x p(t)\, dt$.

This continues problem 7: here, $p(x) = 0.03149\, e^{-x^2/321.1}$, *and*

$$P(x) = \frac{1}{2} + \int_0^x 0.03149\, e^{-t^2/321.1}\, dt.$$

b. [5 points] Write a Taylor series for $p(x)$ (around $x = 0$).

Solution: We know that $e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$. Thus,

$$\begin{aligned} p(x) &= 0.03149\, e^{-x^2/321.1} \\ &= 0.03149\left(1 - \frac{1}{321.1}x^2 + \frac{1}{2!\,(321.1^2)}x^4 + \cdots + \frac{(-1)^n}{n!\,(321.1^n)}x^{2n} + \cdots\right). \end{aligned}$$

c. [5 points] Write a Taylor series for $P(x)$ (around $x = 0$). *Hint: you will probably want to use your work from (b).*

Solution: Using the result from (a) and integrating the series in (b), we have

$$\begin{aligned} P(x) &= \frac{1}{2} + \int_0^x 0.03149 \sum_{n=0}^{\infty} \frac{t^{2n}}{n!\,321.1^n}\, dt \\ &= \frac{1}{2} + 0.03149 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)\,321.1^n}. \end{aligned}$$

d. [4 points] Use your result from (b) or (c) to estimate the probability that there will be up to 60 inches of snow this winter.

Solution: This probability is just $P(60 - 52.91) = P(7.09)$. Using the first two terms of the series in (c), we have $P(7.09) \approx \frac{1}{2} + 0.03149\left(7.09 - \frac{(7.09)^3}{3\,(321.1)}\right) = 0.712$, or about a 71.2% chance. (The leading term gives 72.3%.)
We could also use the series for $p(x)$: $p(x) \approx 0.03149(1 - \frac{1}{321.1}x^2)$, so the probability we want is $\approx \frac{1}{2} + \int_0^{7.09} 0.03149(1 - \frac{1}{321.1}x^2)\,dx$, which will clearly be the same as the preceding. Obviously, trying to use $\int_{-\infty}^{7.09} 0.03149(1 - \frac{1}{321.1}x^2)\,dx$ does not work.

7. [12 points] Determine whether the following series converge or diverge (circle your answer). Be sure to mention which tests you used to justify your answer. If you use the comparison test or limit comparison test, write an appropriate comparison function.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}$

Solution: Since

$$\lim_{n\to\infty}\frac{\sqrt{n}}{1+2\sqrt{n}}=\lim_{n\to\infty}\frac{\sqrt{n}}{2\sqrt{n}}=\frac{1}{2},$$

then the sequence $a_n=(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}$ does not converge to zero (it oscillates closer to $\frac{1}{2}$ and $-\frac{1}{2}$. Since the terms a_n does not converge to 0, then the series $\sum_{n=1}^{\infty}a_n$ diverges.

b. [4 points] $\displaystyle\sum_{n=1}^{\infty}ne^{-n^2}$

Solution: Let $f(x)=xe^{-x^2}$. The function $f(x)>0$ and $f'(x)=e^{-x^2}(1-2x^2)<0$ for $x\geq 1$. Hence by the Integral test

$$\sum_{n=1}^{\infty}ne^{-n^2}\text{ behaves as }\int_1^{\infty}xe^{-x^2}dx=\lim_{b\to\infty}\int_1^b xe^{-x^2}dx=\lim_{b\to\infty}-\frac{1}{2}e^{-x^2}|_1^b=\frac{1}{2e}$$

the series converges.

c. [5 points] $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n^2)}{n^2}$

Solution: This series has positive and negative terms. Since

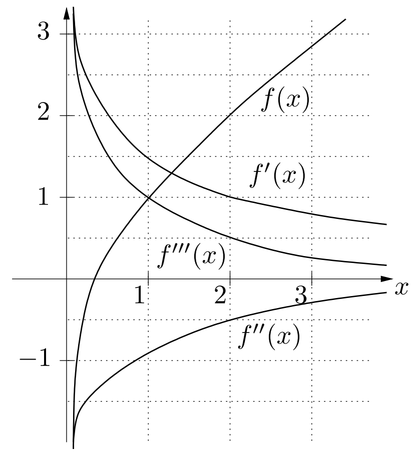
$$\left|\frac{\cos(n^2)}{n^2}\right|\leq\frac{1}{n^2},$$

then the series of the absolute values satisfies

$$\sum_{n=1}^{\infty}\left|\frac{\cos(n^2)}{n^2}\right|\leq\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

The series on the right converges by the p series test with $p=2$, hence the series of absolute values converges. Since the series of absolute values converges, then $\sum_{n=1}^{\infty}\frac{\cos(n^2)}{n^2}$ converges.

2. [7 points] The graph to the right shows $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$.



(a) [4 points of 7] Find the 3rd degree Taylor polynomial approximating $f(x)$ near $x = 2$.

Solution:
We know that the 3rd degree Taylor polynomial is $P_3 = f(2) + f'(2)(x-2) + \frac{1}{2!}f''(2)(x-2)^2 + \frac{1}{3!}f'''(2)(x-2)^3$. We can read the values for f and its derivatives from the graphs, finding $f(x) = 2$, $f'(2) = 1$, $f''(2) = -\frac{1}{2}$ and $f'''(2) = \frac{1}{2}$. Thus

$$P_3 = 2 + (x-2) - \frac{1}{4}(x-2)^2 + \frac{1}{12}(x-2)^3.$$

(b) [3 points of 7] Based on the graphs of f and its derivatives that you have in the given figure, what would you guess the radius of convergence of the Taylor expansion for $f(x)$ around $x = 2$ would be? Explain.

Solution:
From the graphs it is clear that f and its derivatives have a vertical asymptote at $x = 0$. It is not possible for a polynomial expansion to reproduce this, so we would expect that the Taylor expansion would fail there. This is two units from $x = 2$, so we guess that the radius of convergence is $R = 2$.

9. (10 points) At age 65, Mrs. Smith retires with \$1,000,000 in her retirement account. Assume that after retirement:

(i) She receives interest of 5% per year (compounded continuously) on the balance in the account, and this money is reinvested in the account ;

(ii) She withdraws money (for living expenses) from the account at a continuous rate of \$60,000 per year.

(a) Write the initial value problem for the balance $B(t)$ of dollars in the account t years after Mrs. Smith retires.

From the information given in the problem, we deduce that the balance in dollars, $B(t)$, in Mrs. Smith's account satisfies the differential equation

$$B'(t) \;=\; 0.05\,B(t) \;-\; 60,000$$

and has the initial value $B(0) = 1,000,000$ dollars at time 0.

(b) Will Mrs. Smith ever exhaust the retirement account, i.e. reduce the balance in the account to zero? *Explain.*

The amount of money B in the account is being reduced at all times. Indeed, originally $dB/dt = 0.05B - 60,000 < 50,000 - 60,000 < 0$. As B decreases, so does $0.05B$. Hence this also happens at all future times. In particular, the account decreases by at least \$10,000 each year, and it will thus eventually be depleted.

Although the problem does not require it, one can find out when the account will be depleted by solving the equation $B(T) = 0$, for T. The solution to the differential equation in part **(a)** *is $B(t) = 1,200,000 + A\,e^{0.05t}$, where A is a constant. Since $B(0) = 1,000,000$, we get $A = -200,000$ and thus $B(t) = 1,200,000 - 200,000\,e^{0.05t}$. The solution of the equation $B(T) = 0$ is found to be $T = (\ln 6)/0.05 \approx 36$ so it will take about 36 years to deplete the account.*

(c) Are there any equilibrium solutions to the differential equation of part **(a)**? If so, explain their meaning in terms of Mrs. Smith's money.

The only equilibrium solution of the differential equation from part **(a)** *is where $dB/dt = 0$ for all times t or where $B = 60,000/0.05 = 1,200,000$.*

This is the amount of money necessary so that the interest generated is exactly equal to the amount that Mrs. Smith is withdrawing, so that the balance in the account remains constant.

8. [10 points] For $\alpha > 0$, consider the family of spirals given by $r = \frac{1}{\theta^\alpha}$ in polar coordinates.

a. [2 points] Write down an integral that gives the length L of a spiral in this family for $1 \leq \theta \leq b$. No credit will be given if you just write down the formula given in part (b).

Solution: Using parametric equations: $x = \frac{1}{\theta^\alpha}\cos\alpha$ and $y = \frac{1}{\theta^\alpha}\sin\alpha$.

$$L = \int_1^b \sqrt{\left(-\frac{\alpha}{\theta^{\alpha+1}}\cos\theta - \frac{1}{\theta^\alpha}\sin\theta\right)^2 + \left(-\frac{\alpha}{\theta^{\alpha+1}}\sin\theta + \frac{1}{\theta^\alpha}\cos\theta\right)^2}\, d\theta$$

or

$$L = \int_1^b \sqrt{\left(\frac{1}{\theta^\alpha}\right)^2 + \left(-\frac{\alpha}{\theta^{\alpha+1}}\right)^2}\, d\theta$$

b. [8 points] It can be shown that the length L of the spiral in part a) may also be written as

$$L = \int_1^b \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\, d\theta.$$

Use this formula for L to find all values of $\alpha > 0$ for which the length of the spiral is infinite for $1 \leq \theta$. For which values of α is the length finite? Justify all your answers using the comparison test.

Solution:

$$\int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\, d\theta \qquad \text{behaves as} \qquad \int_1^\infty \frac{1}{\theta^\alpha}\, d\theta$$

Since

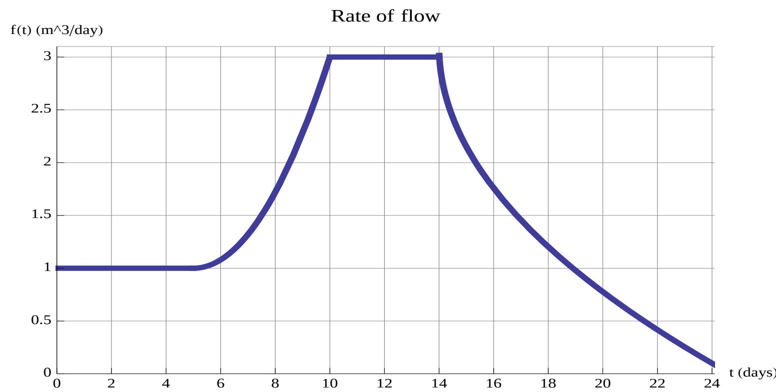
$$\int_1^\infty \frac{1}{\theta^\alpha}\, d\theta \leq \int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\, d\theta$$

then arc length is infinite of $\alpha \leq 1$.

$$\int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\, d\theta \leq \int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1 + \alpha^2}\, d\theta = \sqrt{1 + \alpha^2}\int_1^\infty \frac{1}{\theta^\alpha}\, d\theta$$

Hence the arc length is finite of $\alpha > 1$.

3. [11 points] Sewage flows into the tank described in the previous problem at a rate of $f(t)$ cubic meters per day. Let t be the number of days since December 1, when the tank had 1 m^3 of sewage. A graph of $f(t)$ is given below. Use it to answer the following questions.



a. [3 points] Suppose that $V(t)$ gives the volume of sewage in the tank at time t. Find a formula for $V(t)$ in terms of $f(t)$.

Solution: $V(t) = 1 + \int_0^t f(t)dt$

b. [2 points] For what times t in $[0, 24]$ is $V(t)$ concave up? ______________________

Solution: $4 < t < 10$

c. [2 points] For what times t in $[0, 24]$ is $V(t)$ concave down? ______________________

Solution: $14 < t < 24$

d. [4 points] Fill out the table below. Using the values in your table, compute Riemann sums with 3 subintervals to find an underestimate and an overestimate for $V(12)$. Justify why the Riemann sums you selected yield the appropriate under and upper estimates. Do not forget to include the units in your answer.

t	0	4	8	12
$f(t)$				

Solution:

t	0	4	8	12
$f(t)$	1	1	$\approx$ 1.75	3

$f(t)$ is increasing in $(0, 12)$ then
Upper estimate (Right hand sum): $4(1 + 1.75 + 3) + 1 = 24\ m^3$
Lower estimate (Left hand sum): $4(1 + 1 + 1.75) + 1 = 16\ m^3$

1. [12 points] While at home for Thanksgiving, Alex finds a forgotten can of corn that has been sitting on the shelf for a number of years. The contents have started to settle towards the bottom of the can, and the density of corn inside the can is therefore a function, $\delta(h)$, of the height h (measured in cm) from the bottom of the can. δ is measured in g/cm^3. The can has a radius of 4 cm, and a height of 12 cm.

(a) [3 points of 12] Write an expression that approximates the mass of corn in the cylindrical cross-section from height h to height $h + \Delta h$.

Solution:
The cylindrical cross-section is a disk with height Δh and radius 4 cm, so its volume is $\Delta V = \pi(4)^2 \cdot \Delta h$. The mass of the cross-section is then $\Delta M = \delta(h) \cdot \Delta V = 16\pi \cdot \delta(h) \cdot \Delta h$.

(b) [3 points of 12] Write a definite integral that gives the total mass of corn in the can.

Solution:
We let $\Delta h \to 0$ and add the contributions from each disk by integrating, to get

$$M = \int_0^{12} 16\pi \cdot \delta(h)\, dh.$$

(c) [3 points of 12] If $\delta(h) = 4e^{-0.03h}$, what is the total mass of corn inside the can?

Solution:
We have $M = \int_0^{12} 16\pi \cdot \delta(h)\, dh = \int_0^{12} 16\pi \cdot 4e^{-0.03h}\, dh$. Thus

$$M = 64\pi \int_0^{12} e^{-0.03h}\, dh = -\frac{64}{0.03}\pi\, e^{-0.03h}\Big|_0^{12} = -\frac{6400}{3}\pi\left(e^{-0.36} - 1\right) \approx 2026.2 \text{ g}.$$

(d) [3 points of 12] Write, but do not evaluate, an expression for the can's center of mass in the h direction. Would you expect the center of mass to be in the top or bottom half of the can? Do not solve for the center of mass, but in one sentence, justify your answer.

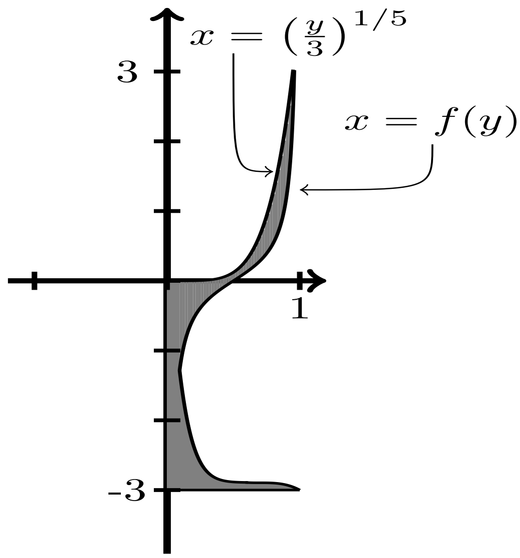
Solution:
The center of mass is

$$\overline{h} = \frac{\int_0^{12} 16\pi \cdot h \cdot \delta(h)\, dh}{M},$$

where M is the mass we found before. We expect this to be in the bottom half of the can, because the density decreases with increasing h. (Obviously, we could also write $\overline{h} = \frac{\int_0^{12} 16\pi \cdot h \cdot \delta(h)\, dh}{\int_0^{12} 16\pi \cdot \delta(h)}$; plugging in $\delta(h)$ from (c) is fine too.)

7. [12 points]

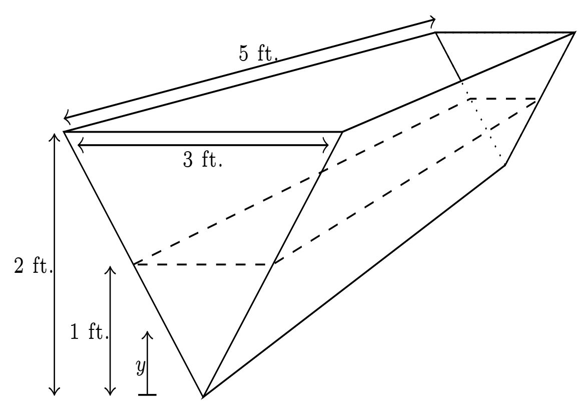
a. [5 points] You rotate the region shown about the y-axis to create a drinking glass. Write an expression that represents the volume of material required to construct the drinking glass (your answer may contain $f(y)$).



Solution:

$$\text{Volume} = \int_0^3 \pi \left(f(y)^2 - \left(\frac{y}{3}\right)^{\frac{2}{5}} \right) dy + \int_{-3}^0 \pi f(y)^2 dy.$$

b. [7 points] Consider the vessel shown below. It is filled to a depth of 1 foot of water. Write an integral in terms of y (the distance in ft from the bottom of the vessel) for the work required to pump all the water to the top of the vessel. Water weighs 62.4 lbs/ft^3.



Solution: Using similar triangles:
Volume of a slice=$5\left(\frac{3}{2}y\right)\Delta y$
Work=$\int_0^1 5\left(\frac{3}{2}y\right)(62.4)(2-y)dy = \int_0^1 468y(2-y)dy$.

8. [13 points] Let $C(u)$ be a function that satisfies $C'(u) = \frac{\cos(u^2)}{u}$, $C(2) = 3$, and let $S(u)$ be a function that satisfies $S'(u) = \frac{\sin(u^2)}{u}$, $S(2) = -1$.

a. [4 points] Write expressions for $C(t)$ and $S(t)$ that satisfy the above conditions.

Solution:

$$C(t) = \int_2^t \frac{\cos(u^2)}{u} du + 3 \qquad S(t) = \int_2^t \frac{\sin(u^2)}{u} du - 1$$

b. [5 points] A particle traces out the curve given by the parametric equations $x(t) = C(\ln(t))$, $y(t) = S(\ln(t))$ for $t \geq 10$. What is the speed of the particle at time t? You may assume that $t \geq 10$.

Solution:

$$\begin{aligned} \text{speed} &= \sqrt{\left(\frac{d}{dt}\int_2^{\ln(t)} \frac{\cos(u^2)}{u} du\right)^2 + \left(\frac{d}{dt}\int_2^{\ln(t)} \frac{\sin(u^2)}{u} du\right)^2} \\ &= \sqrt{\left(\frac{1}{t} \cdot \frac{\cos(\ln(t))^2}{\ln(t)}\right)^2 + \left(\frac{1}{t} \cdot \frac{\sin(\ln(t))^2}{\ln(t)}\right)^2} \\ &= \frac{1}{t\ln(t)} \end{aligned}$$

c. [4 points] For $t \geq 10$, is the curve given by the parametric equations in part (b) of finite or infinite length? Justify your answer.

Solution: The curve has infinite length. The arc length for $t \geq 10$ is given by integrating the speed over the interval $[10, \infty)$. With a u-substitution $u = \ln(t)$ we have

$$\text{arc length} = \int_{10}^{\infty} \frac{dt}{t\ln(t)} = \int_{\ln(10)}^{\infty} \frac{du}{u}$$

which diverges by the p-test.