Math 116 — Practice for Exam 3

Generated April 15, 2012

Name: ____

Instructor: ____________________ Section Number: __________

1. This exam has 13 pages including this cover. There are 10 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2008	3	6	ibuprofin	12	
Fall 2011	3	7		12	
Fall 2008	3	7	snow	16	
Fall 2006	3	2		7	
Fall 2004	3	10	Doppler	8	
Fall 2011	2	8	spirals	10	
Fall 2010	3	3	sewage flow	11	
Fall 2006	2	1	corn	12	
Winter 2011	3	7	wine glass	12	
Fall 2009	1	8		13	
Total				113	

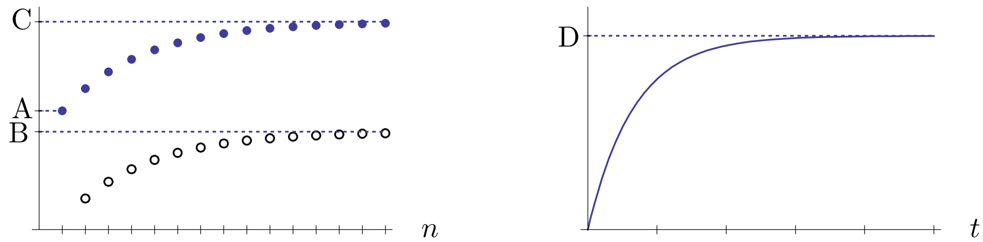
Recommended time (based on points): 125 minutes

6. [12 points] When a patient takes a drug (e.g., by ingesting a pill), the amount of the drug in her/his system changes with time. We can think of this process discretely (each pill is an immediately delivered dose) or continuously (each pill delivers a small amount of drug per unit time over a long time). This problem considers these two different models.

a. [4 points] Suppose that ibuprofen is taken in 200 mg doses every six hours, and that all 200 mg are delivered to the patient's body immediately when the pill is taken. After six hours, 12.5% of the ibuprofen remains. Find expressions for the amount of ibuprofen in the patient immediately before and after the nth pill taken. Include work; without work, you may receive no credit.

b. [4 points] Now suppose that ibuprofen is taken in a time-release capsule that continuously releases 35 mg/hr of ibuprofen per hour for six hours. The drug decays at a rate proportional to the amount in the body, with a constant of proportionality $r = 0.35$. Write a differential equation for the amount of ibuprofen, $y(t)$, in the patient as a function of time. Solve your differential equation, assuming that there is no ibuprofen in the patient initially.

c. [4 points] *This is the third part of problem (6), which begins on the previous page.* The figure below shows graphs that may correspond to the solutions that you obtained in (6a) and (6b). The graph to the left corresponds to part (a), with the solid and open circles in the graph showing the amount of ibuprofen in the patient after and before the nth pill. The graph to the right corresponds to (b). Assuming that these graphs do represent your solutions, use your results from (a) and (b) to determine the values of A, B, C, and D.



7. [16 points] The annual snowfall in Ann Arbor has mean 52.91 inches and standard deviation 12.67 inches. Assume that these data are normally distributed, and let x be the deviation of a year's snowfall from the mean (so that if $x = -2$ in a given year, it means that the snowfall that year was 50.91 inches). Then the probability density function for x is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/(2\sigma^2)} = 0.03149\, e^{-x^2/321.1},$$

so that its cumulative distribution function $P(x)$ is

$$P(x) = \int_{-\infty}^{x} p(t)\, dt = \int_{-\infty}^{x} 0.03149\, e^{-t^2/321.1}\, dt.$$

a. [2 points] Explain why $P(x) = \dfrac{1}{2} + \displaystyle\int_0^x 0.03149\, e^{-t^2/321.1}\, dt.$

This continues problem 7: here, $p(x) = 0.03149\, e^{-x^2/321.1}$, *and*

$$P(x) = \frac{1}{2} + \int_0^x 0.03149\, e^{-t^2/321.1}\, dt.$$

b. [5 points] Write a Taylor series for $p(x)$ (around $x = 0$).

c. [5 points] Write a Taylor series for $P(x)$ (around $x = 0$). *Hint: you will probably want to use your work from (b).*

d. [4 points] Use your result from (b) or (c) to estimate the probability that there will be up to 60 inches of snow this winter.

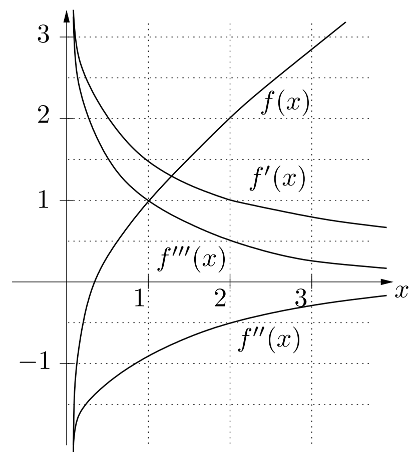
7. [12 points] Determine whether the following series converge or diverge (circle your answer). Be sure to mention which tests you used to justify your answer. If you use the comparison test or limit comparison test, write an appropriate comparison function.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}$

b. [4 points] $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

c. [5 points] $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n^2)}{n^2}$

2. [7 points] The graph to the right shows $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$.



(a) [4 points of 7] Find the 3rd degree Taylor polynomial approximating $f(x)$ near $x = 2$.

(b) [3 points of 7] Based on the graphs of f and its derivatives that you have in the given figure, what would you guess the radius of convergence of the Taylor expansion for $f(x)$ around $x = 2$ would be? Explain.

10. (8 points) We shall investigate a well-known physical phenomenon, called the "Doppler Effect". When an electromagnetic signal (e.g. a ray of light) with frequency F_e is emitted from a source moving away with velocity $v > 0$ with respect to a receiver at rest, then the received frequency F_r is different from F_e. The relationship linking the emitted frequency F_e and the received frequency F_r is the Doppler Law:

$$F_r = \sqrt{\frac{1 - v/c}{1 + v/c}}\, F_e \,, \qquad \text{where } c \text{ is a constant, the speed of light.}$$

For this problem, you might find useful to know that the Taylor series for the function $\sqrt{\dfrac{1+x}{1-x}}$ near $x = 0$ is $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{2} + \cdots$.

(a) On Earth, nearly all objects travel with velocities v much smaller than the speed of light c, i.e. the ratio v/c is very small. Use this fact to obtain the approximation to the Doppler Law for slow-moving emitters:

$$F_r \simeq \left(1 - \frac{v}{c}\right) F_e \,.$$

(b) The relationship in part **(a)** is *not* exact, and an error is made when it is used to approximate the Doppler Law. Find an expression for the "error", in terms of v, c and F_e. Is the approximation accurate within 1% of F_e if the velocity is at most 20% of the speed of light c? *Explain.*

8. [10 points] For $\alpha > 0$, consider the family of spirals given by $r = \frac{1}{\theta^\alpha}$ in polar coordinates.

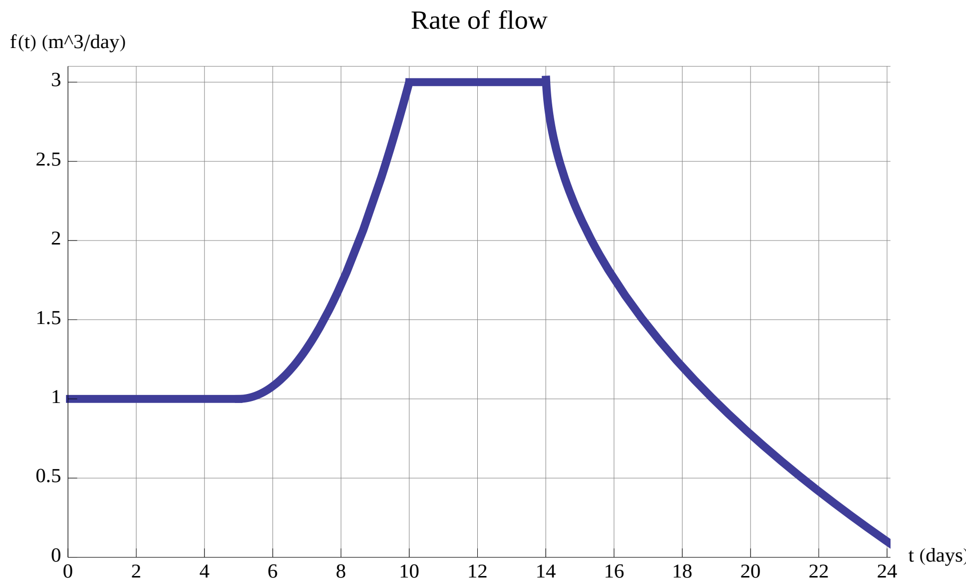
a. [2 points] Write down an integral that gives the length L of a spiral in this family for $1 \le \theta \le b$. No credit will be given if you just write down the formula given in part (b).

b. [8 points] It can be shown that the length L of the spiral in part a) may also be written as

$$L = \int_1^b \frac{1}{\theta^\alpha}\sqrt{1+\frac{\alpha^2}{\theta^2}}\,d\theta.$$

Use this formula for L to find all values of $\alpha > 0$ for which the length of the spiral is infinite for $1 \le \theta$. For which values of α is the length finite? Justify all your answers using the comparison test.

3. [11 points] Sewage flows into the tank described in the previous problem at a rate of $f(t)$ cubic meters per day. Let t be the number of days since December 1, when the tank had 1 m^3 of sewage. A graph of $f(t)$ is given below. Use it to answer the following questions.



a. [3 points] Suppose that $V(t)$ gives the volume of sewage in the tank at time t. Find a formula for $V(t)$ in terms of $f(t)$.

b. [2 points] For what times t in $[0, 24]$ is $V(t)$ concave up? ____________________

c. [2 points] For what times t in $[0, 24]$ is $V(t)$ concave down? ____________________

d. [4 points] Fill out the table below. Using the values in your table, compute Riemann sums with 3 subintervals to find an underestimate and an overestimate for $V(12)$. Justify why the Riemann sums you selected yield the appropriate under and upper estimates. Do not forget to include the units in your answer.

t	0	4	8	12
$f(t)$				

1. [12 points] While at home for Thanksgiving, Alex finds a forgotten can of corn that has been sitting on the shelf for a number of years. The contents have started to settle towards the bottom of the can, and the density of corn inside the can is therefore a function, $\delta(h)$, of the height h (measured in cm) from the bottom of the can. δ is measured in g/cm^3. The can has a radius of 4 cm, and a height of 12 cm.

(a) [3 points of 12] Write an expression that approximates the mass of corn in the cylindrical cross-section from height h to height $h + \Delta h$.

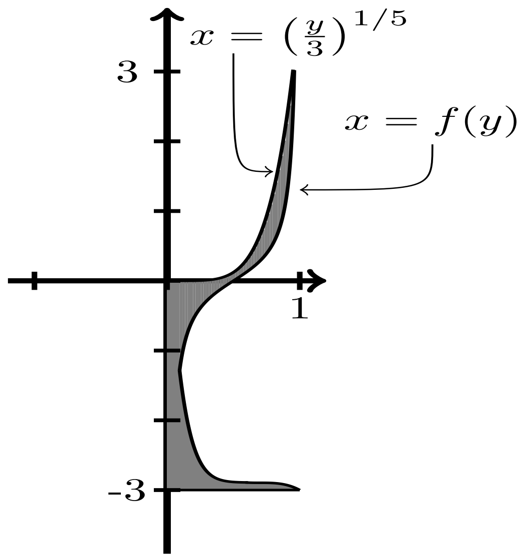
(b) [3 points of 12] Write a definite integral that gives the total mass of corn in the can.

(c) [3 points of 12] If $\delta(h) = 4e^{-0.03h}$, what is the total mass of corn inside the can?

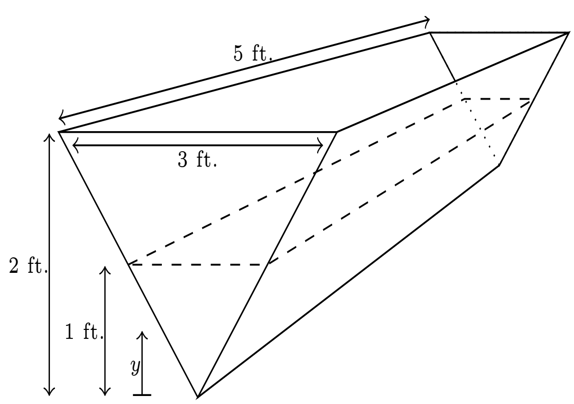
(d) [3 points of 12] Write, but do not evaluate, an expression for the can's center of mass in the h direction. Would you expect the center of mass to be in the top or bottom half of the can? Do not solve for the center of mass, but in one sentence, justify your answer.

7. [12 points]

a. [5 points] You rotate the region shown about the y-axis to create a drinking glass. Write an expression that represents the volume of material required to construct the drinking glass (your answer may contain $f(y)$).



b. [7 points] Consider the vessel shown below. It is filled to a depth of 1 foot of water. Write an integral in terms of y (the distance in ft from the bottom of the vessel) for the work required to pump all the water to the top of the vessel. Water weighs 62.4 lbs/ft^3.



8. [13 points] Let $C(u)$ be a function that satisfies $C'(u) = \frac{\cos(u^2)}{u}$, $C(2) = 3$, and let $S(u)$ be a function that satisfies $S'(u) = \frac{\sin(u^2)}{u}$, $S(2) = -1$.

a. [4 points] Write expressions for $C(t)$ and $S(t)$ that satisfy the above conditions.

b. [5 points] A particle traces out the curve given by the parametric equations $x(t) = C(\ln(t))$, $y(t) = S(\ln(t))$ for $t \geq 10$. What is the speed of the particle at time t? You may assume that $t \geq 10$.

c. [4 points] For $t \geq 10$, is the curve given by the parametric equations in part (b) of finite or infinite length? Justify your answer.