A Refinement of the Eulerian numbers, and the Joint Distribution of $\pi(1)$ and $\mathrm{Des}(\pi)$ in S_n

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Abstract

Given a permutation π chosen uniformly from S_n, we explore the joint distribution of $\pi(1)$ and the number of descents in π. We obtain a formula for the number of permutations with $\mathrm{Des}(\pi) = d$ and $\pi(1) = k$, and use it to show that if $\mathrm{Des}(\pi)$ is fixed at d, then the expected value of $\pi(1)$ is $d+1$. We go on to derive generating functions for the joint distribution, show that it is unimodal if viewed correctly, and show that when d is small the distribution of $\pi(1)$ among the permutations with d descents is approximately geometric. Applications to Stein's method and the Neggers-Stanley problem are presented.

1 Introduction

Consider S_n to be the set of all bijections from $\{1, 2, \ldots, n\}$ to itself. We will often identify a permutation π with the sequence $\pi(1), \pi(2), \ldots, \pi(n)$. So for instance if $\pi(1) = k$ and $\pi(n) = \ell$, we say that π "begins with" k and "ends with" ℓ.

A permutation π is said to have a descent at i if $\pi(i) > \pi(i+1)$. That is to say, if we graph the points $(i, \pi(i))$ and connect them left to right, descents are the positions at which the connecting segments have negative slope. Let $\mathrm{Des}(\pi)$ be the number of descents in π, and define

$$(1) \qquad \left\langle {n \atop d} \right\rangle := \# \left\{ \pi \in S_n : \mathrm{Des}(\pi) = d \right\}.$$

These are known as the Eulerian numbers, and have been widely studied; see, for example, [GKP94, p. 267] and [Car59].

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Bayer and Diaconis [BD92] showed that the probability that a particular permutation of a deck of cards occurs after any number of riffle shuffles is determined by the number of descents the permutation has. In [CVar], Viswanath and the author began working to generalize that result to decks containing repeated cards. At one point we had occasion to consider the number of permutations of n letters which have d descents and begin with k. That is,

$$\left\langle {n \atop d} \right\rangle_k := \#\left\{\pi \in S_n : \mathrm{Des}(\pi) = d \text{ and } \pi(1) = k\right\}. \tag{2}$$

The current work is an investigation of the numbers defined in Equation (2). We derive a formula in terms of binary coefficients:

Theorem 1 *If $1 \le k \le n$,*

$$\left\langle {n \atop d} \right\rangle_k = \sum_{j \ge 0} (-1)^{d-j} \binom{n}{d-j} j^{k-1} (j+1)^{n-k}$$

where 0^0 is interpreted as 1

which is similar to a well-known formula for the Eulerian numbers. We use the formula to understand how the two statistics $\mathrm{Des}(\pi)$ and $\pi(1)$ interact.

If we are constructing a permutation with d descents from left to right, and d is small, a conservative strategy would seem to be to start with a low number, since starting with a high number means we will use up one of our descents near the beginning of the permutation. So in other words, we expect that if d is small then there are more permutations with d descents starting with low numbers than starting with high numbers. Similarly, if d is close to n, our intuition is that that starting with a high number leaves us more possibilities later on. This intuition turns into a surprisingly simple result:

Theorem 3 *If π is chosen uniformly from among those permutations of n that have d descents, the expected value of $\pi(1)$ is $d+1$ and the expected value of $\pi(n)$ is $n-d$.*

And in Theorem 7 we find, as expected, that the sequence

$$\left\langle {n \atop d} \right\rangle_1, \left\langle {n \atop d} \right\rangle_2, \ldots, \left\langle {n \atop d} \right\rangle_n$$

is weakly decreasing when d is small and weakly increasing when d is large. Consequently that sequence is an interpolation between its endpoints, which are two Eulerian numbers: $\left\langle {n-1 \atop d} \right\rangle$ and $\left\langle {n-1 \atop d-1} \right\rangle$. Experimental evidence (see Section 10) suggests that it is a good interpolation, at least when d is close to $(n-1)/2$, in the sense that a normal approximation to the Eulerian numbers also seems to provide a good approximation to the refined Eulerian numbers. However, the normal approximation is good for neither set when d is small or d

is close to n. Theorem 7 shows that in those cases the distribution of $\pi(1)$ is approximately geometric.

The application which led directly to the current work is presented in Section 5. Fulman shows in [Ful04] that certain statistics on permutations, one of which is descents, are approximately normally distributed. The main tool he uses is Stein's method, due to Charles Stein in [Ste86]. The thrust behind the method is to introduce a little extra randomness to a given random variable to get a new one. If certain symmetries are present, the result is an "exchangeable pair" of random variables, meaning, essentially, that the Markov process which takes one to the other is reversible. Then Stein's theorems (and more recent refinements of them) can be applied to bound the distance between the original variable's distribution and the standard normal distribution.

Fulman uses a "random to end" operation to add randomness to permutations. That is, he starts with a uniformly distributed permutation π and sets

$$\pi' = (I, I+1, \ldots, n)\pi$$

where I is selected uniformly from $\{1, 2, \ldots, n\}$. While (π, π') is not an exchangeable pair, it turns out that $(\mathrm{Des}(\pi), \mathrm{Des}(\pi'))$ is, and this leads to a central limit theorem for descents, and for a whole class of statistics.

We tried a different method of adding randomness to π, namely, following π by a uniformly selected transposition. That calculation (which is presented in Section 5) led directly to Theorem 3.

The Neggers-Stanley Conjecture, now proved false in general ([Brä04, Stear]), was that the generating function for descents among the linear extensions of any poset has only real zeroes. Since a function with positive coefficients can have no positive zeroes, any combinatorial generating function with all real zeroes can be written in the form

$$a(x + c_1)(x + c_2) \cdots (x + c_n)$$

for non-negative constants $a, c_1, c_2, \ldots, c_n$. The implication, then, is that if D is the number of descents in a uniformly selected linear extension of a poset for which the Neggers-Stanley conjecture is true, then D can be written as the sum of independent Bernoulli variables.

In Section 6 we present several generating functions for the refined Eulerian numbers. The set of permutations of n which begin with k is the same as the set of linear extensions of the poset defined on $\{1, 2, \ldots, n\}$ by $k < a$ for all a other than k. So we can find the Neggers-Stanley generating function for this poset explicitly, and we show that it does indeed have only real zeroes. We go on to show that several similar posets also satisfy the conjecture. (All of the posets considered were known to satisfy the conjecture by theorems of Simion [Sim84] and Wagner [Wag92].)

2 Basic Properties

If $\pi(1) = 1$, then $\pi(1)$ is certainly less than $\pi(2)$, so all descents are among the final $n-1$ numbers. And if $\pi(1) = n$, there is certain to be a descent between $\pi(1)$ and $\pi(2)$. So we know some boundary values:

$$\left\langle\begin{matrix} n \\ d \end{matrix}\right\rangle_1 = \left\langle\begin{matrix} n-1 \\ d \end{matrix}\right\rangle \quad \text{and} \quad \left\langle\begin{matrix} n \\ d \end{matrix}\right\rangle_n = \left\langle\begin{matrix} n-1 \\ d-1 \end{matrix}\right\rangle \tag{3}$$

for $n > 1$. Also, it is immediate that

$$\sum_d \left\langle\begin{matrix} n \\ d \end{matrix}\right\rangle_k = (n-1)! \tag{4}$$

$$\sum_k \left\langle\begin{matrix} n \\ d \end{matrix}\right\rangle_k = \left\langle\begin{matrix} n \\ d \end{matrix}\right\rangle. \tag{5}$$

Let $\rho \in S_n$ be the reversal permutation: $\rho(i) = n+1-i$. Then $\rho\pi$ is the same as π but with i replaced by $n+1-i$ everywhere. As a result, $\rho\pi$ has a descent wherever π has an ascent, and an ascent wherever π has a descent. So $\mathrm{Des}(\rho\pi) = n-1-\mathrm{Des}(\pi)$. Since $\pi \mapsto \rho\pi$ is a bijection from S_n to itself, it follows that

$$\left\langle\begin{matrix} n \\ d \end{matrix}\right\rangle = \left\langle\begin{matrix} n \\ n-1-d \end{matrix}\right\rangle. \tag{6}$$

Note we could have obtained the same result from the map $\pi \mapsto \pi\rho$, since reversing π changes ascents to descents and also reflects their positions about the center.

Let

$$\left\langle\begin{matrix} n \\ d \end{matrix}\right\rangle^k := \#\left\{\pi \in S_n : \mathrm{Des}(\pi) = d \text{ and } \pi(n) = k\right\}. \tag{7}$$

Both transformations yield symmetric identities for the refined Eulerian numbers. If

$$\pi(1) = k \quad \text{and} \quad \mathrm{Des}(\pi) = d$$

then

$$\begin{aligned} \rho\pi(1) = n+1-k \quad &\text{and} \quad \mathrm{Des}(\rho\pi) = n-1-d \\ \pi\rho(n) = k \quad &\text{and} \quad \mathrm{Des}(\pi\rho) = n-1-d \\ \rho\pi\rho(n) = n+1-k \quad &\text{and} \quad \mathrm{Des}(\rho\pi\rho) = d \end{aligned}$$

from which it follows that

$$\left\langle\begin{matrix} n \\ d \end{matrix}\right\rangle_k = \left\langle\begin{matrix} n \\ n-1-d \end{matrix}\right\rangle_{n+1-k} = \left\langle\begin{matrix} n \\ n-1-d \end{matrix}\right\rangle^k = \left\langle\begin{matrix} n \\ d \end{matrix}\right\rangle^{n+1-k}. \tag{8}$$

3 Recurrences

Assume $n > 1$. Let

$$T_k := \{\pi \in S_n : \pi(1) = k \text{ and } \mathrm{Des}(\pi) = d\}$$
$$T_{k,\ell} := \{\pi \in S_n : \pi(1) = k,\ \pi(2) = \ell, \text{ and } \mathrm{Des}(\pi) = d\}$$

and let $\pi \in T_{k,\ell}$. If $\ell < k$, then there is a descent between $\pi(1)$ and $\pi(2)$, so there must be $d-1$ descents in the "tail", $\pi(2), \pi(3), \ldots, \pi(n)$. The tail begins with ℓ, which is the ℓth largest value in the tail, so we must have

$$\#T_{k,\ell} = \left\langle {n-1 \atop d-1} \right\rangle_\ell$$

when $\ell < k$. Likewise, if $\ell > k$, there is no descent between $\pi(1)$ and $\pi(2)$, so there must be d descents in the tail. This time ℓ is the $(\ell-1)$st largest value in the tail, so

$$\#T_{k,\ell} = \left\langle {n-1 \atop d} \right\rangle_{\ell-1}$$

when $\ell > k$. Of course T_k is the disjoint union of the $T_{k,l}$, so

$$\left\langle {n \atop d} \right\rangle_k = \#T_k = \sum_\ell \#T_{k,\ell} = \sum_{\ell<k} \left\langle {n-1 \atop d-1} \right\rangle_\ell + \sum_{\ell>k} \left\langle {n-1 \atop d} \right\rangle_{\ell-1}$$

or, more succinctly,

(9) $$\left\langle {n \atop d} \right\rangle_k = \sum_{\ell=1}^{n-1} \left\langle {n-1 \atop d-[\ell<k]} \right\rangle_\ell$$

where the bracket notation follows [GKP94]: $[A]$ is 1 if A is true and 0 if A is false. (Knuth refers to this as Iverson notation in [Knu92], and traces its origin to [Ive62].) Note Equation (9) fails when $k < 1$ or $k > n$, in which case $\left\langle {n \atop d} \right\rangle_k = 0$.

Now suppose $1 \le k \le n-1$ and $\pi \in S_n$ begins with k. Swapping k with $k+1$ in the sequence $\pi(1), \pi(2), \ldots, \pi(n)$ preserves descents for most π; the only exception is when $\pi(2) = k+1$, in which case a new descent is created. If we eliminate that case, the swap map is a bijection from $T_k \setminus T_{k,k+1}$ to $T_{k+1} \setminus T_{k+1,k}$, as those sets are defined above. Substituting sizes for sets, we have

(10) $$\left\langle {n \atop d} \right\rangle_k - \left\langle {n-1 \atop d} \right\rangle_k = \left\langle {n \atop d} \right\rangle_{k+1} - \left\langle {n-1 \atop d-1} \right\rangle_k.$$

Equation (10) is valid as long as $k \neq 0$ and $k \neq n$. (If $k < 0$ or $k > n$, all terms are 0.)

A well-know recurrence for $\left\langle {n \atop d} \right\rangle$ comes from considering what happens when you insert n into an element of S_{n-1}:

(11) $$\left\langle {n \atop d} \right\rangle = (n-d)\left\langle {n-1 \atop d-1} \right\rangle + (d+1)\left\langle {n-1 \atop d} \right\rangle$$

We can get a similar recurrence for the refined Eulerian numbers by considering what happens when you insert n into an element of S_{n-1} which begins with k:

$$\left\langle {n \atop d} \right\rangle_k = (n-d-1)\left\langle {n-1 \atop d-1} \right\rangle_k + (d+1)\left\langle {n-1 \atop d} \right\rangle_k. \tag{12}$$

In other words, one way to get an element of S_n which begins with k and has d descents is to take an element of S_{n-1} which begins with k and has d descents, and insert n at a descent or at the end ($d+1$ choices). The other way is to start with an element of S_{n-1} which begins with k and has $d-1$ descents, and insert n at an ascent ($n-d-1$ choices). Equation (12) fails when $k = n$, since a permutation of S_{n-1} cannot begin with n. It is valid for all other values of k.

4 Formulas and Moments

There is an explicit formula for the Eulerian numbers in terms of binomial coefficients:

$$\left\langle {n \atop d} \right\rangle = \sum_{j \geq 0} (-1)^{d-j} \binom{n+1}{d-j} (j+1)^n. \tag{13}$$

See for example, [GKP94, p. 269]. [Aside: Equation (13) follows from Equation (11), which means that it is valid for all values of d, even if $d < 0$ or $d \geq n$]. So we have

$$\left\langle {n \atop d} \right\rangle_1 = \left\langle {n-1 \atop d} \right\rangle = \sum_{j \geq 0} (-1)^{d-j} \binom{n}{d-j} (j+1)^{n-1} \tag{14}$$

$$\left\langle {n \atop d} \right\rangle_n = \left\langle {n-1 \atop d-1} \right\rangle = \sum_{j \geq 0} (-1)^{d-1-j} \binom{n}{d-1-j} (j+1)^{n-1} = \sum_{j \geq 0} (-1)^{d-j} \binom{n}{d-j} j^{n-1}. \tag{15}$$

These suggest a formula for $\left\langle {n \atop d} \right\rangle_k$:

Theorem 1 *If $1 \leq k \leq n$,*

$$\left\langle {n \atop d} \right\rangle_k = \sum_{j \geq 0} (-1)^{d-j} \binom{n}{d-j} j^{k-1} (j+1)^{n-k} \tag{16}$$

where 0^0 is interpreted as 1.

Proof. Fix $k \geq 1$. The theorem is true for $n = k$ by Equation (15). Suppose it is true for some $n \geq k$. Then

$$\begin{aligned} \left\langle {n+1 \atop d} \right\rangle_k &= (n-d)\left\langle {n \atop d-1} \right\rangle_k + (d+1)\left\langle {n \atop d} \right\rangle_k \\ &= \sum_{j \geq 0} (-1)^{d-j} \left[-(n-d)\binom{n}{d-1-j} + (d+1)\binom{n}{d-j} \right] j^{k-1}(j+1)^{n-k}. \end{aligned}$$

The quantity in brackets reduces to $(j+1)\binom{n+1}{d-j}$, so the theorem is true for all $n \geq k$ by induction. □

Note that we assumed nothing about d; Equation (16) is valid even if $d < 0$ or $d \geq n$.

From Equation (16) we can deduce a formula for the mth "rising moment" of $\pi(1)$ when $\mathrm{Des}(\pi)$ is fixed. Assume π is chosen uniformly from S_n, and let

$$\mu_m := \mathbb{E}^{\mathrm{Des}(\pi)=d}\pi(1)^{\overline{m}} \tag{17}$$

where $x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1)$.

Lemma 2

$$\left\langle {n \atop d} \right\rangle \mu_m = m! \sum_{j\geq 0}(-1)^{d-j}\binom{n}{d-j}\sum_{\ell=0}^{n-1}\binom{m+n}{\ell} j^{\ell}. \tag{18}$$

Proof. From Equation (16),

$$\begin{aligned}\left\langle {n \atop d} \right\rangle \mu_m &= \sum_{k=1}^{n} k^{\overline{m}} \left\langle {n \atop d} \right\rangle_k = \sum_{k=1}^{n}\frac{(k+m-1)!}{(k-1)!}\sum_{j\geq 0}(-1)^{d-j}\binom{n}{d-j} j^{k-1}(j+1)^{n-k} \\ &= m!\sum_{j\geq 0}(-1)^{d-j}\binom{n}{d-j}\sum_{r=0}^{n-1}\binom{r+m}{r} j^r (j+1)^{n-1-r}\end{aligned}$$

(the last by setting $r = k-1$). But $(j+1)^{n-1-r} = \sum_{s=0}^{n-1-r}\binom{n-1-r}{s} j^s$. So let $\ell = r+s$ and we have

$$\left\langle {n \atop d} \right\rangle \mu_m = m!\sum_{j\geq 0}(-1)^{d-j}\binom{n}{d-j}\sum_{\ell=0}^{n-1} j^{\ell}\sum_{r=0}^{\ell}\binom{r+m}{r}\binom{n-1-r}{\ell-r}. \tag{19}$$

Let ϕ be a north/east lattice path from $(0,0)$ to $(m+n-\ell, \ell)$ (see Figure 1). The number of such paths is $\binom{m+n}{\ell}$. If r is the height at which ϕ crosses the line $x = m+\frac{1}{2}$, then ϕ consists of a path from $(0,0)$ to (m,r), a horizontal segment, and a path from $(m+1,r)$ to $(m+n-\ell,\ell)$. Counting the possibilities for the parts yields the identity

$$\sum_{r=0}^{\ell}\binom{r+m}{r}\binom{n-1-r}{\ell-r} = \binom{m+n}{\ell}. \tag{20}$$

Substituting Equation (20) into Equation (19) yields the desired result. □

Note that the last sum in Equation (18) is a truncated binomial expansion of $(j+1)^{m+n}$.

Theorem 3 *If π is chosen uniformly from among those permutations of n that have d descents, the expected value of $\pi(1)$ is $d+1$ and the expected value of $\pi(n)$ is $n-d$.*

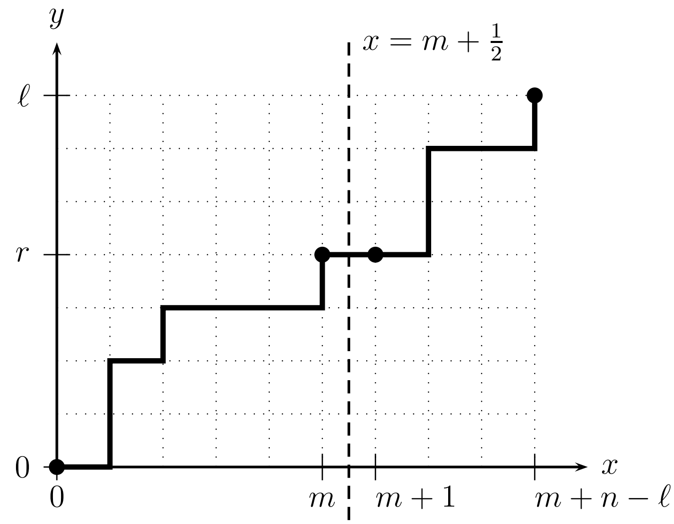


Figure 1: A north-east lattice path from $(0,0)$ to $(m+n-\ell,\ell)$. (All edges are either north or east.)

Proof. The expected value of $\pi(1)$ is μ_1, and

$$\begin{aligned}
\left\langle {n \atop d} \right\rangle \mu_1 &= \sum_{j\geq 0} (-1)^{d-j} \binom{n}{d-j} \sum_{\ell=0}^{n-1} \binom{n+1}{\ell} j^\ell \\
&= \sum_{j\geq 0} (-1)^{d-j} \binom{n}{d-j} \left((j+1)^{n+1} - j^{n+1} - (n+1)j^n\right) \\
&= \sum_{j\geq 0} (-1)^{d-j} \binom{n}{d-j} (j+1)^{n+1} - \sum_{i\geq 0} (-1)^{d-i} \binom{n}{d-i} (n+1+i) i^n.
\end{aligned}$$

The term for $i = 0$ is 0, so let $j = i - 1$ and combine

$$= \sum_{j\geq 0} (-1)^{d-j} (j+1)^n \left[\binom{n}{d-j} (j+1) + \binom{n}{d-j-1} (n+j+2) \right].$$

The quantity in brackets simplifies to $(d+1)\binom{n+1}{d-j}$, so

$$\left\langle {n \atop d} \right\rangle \mu_1 = (d+1) \sum_{j\geq 0} (-1)^{d-j} (j+1)^n \binom{n+1}{d-j} = (d+1) \left\langle {n \atop d} \right\rangle.$$

Therefore

$$\mu_1 = \mathbb{E}^{\mathrm{Des}(\pi)=d} \pi(1) = d+1.$$

For the second part,

$$\begin{aligned}
\mathbb{E}^{\mathrm{Des}(\pi)=d} \pi(n) &= \frac{1}{n!} \sum_k k \left\langle {n \atop d} \right\rangle^k = \frac{1}{n!} \sum_k k \left\langle {n \atop n-1-d} \right\rangle_k \\
&= \mathbb{E}^{\mathrm{Des}(\pi)=n-1-d} \pi(1) = n - d.
\end{aligned}$$

□

5 Application Using Stein's Method

Charles Stein developed a method for showing that the distribution of a random variable W which meets certain criteria is approximately standard normal. His technique has come to be known as Stein's method; see [Ste86] or [DH04] for more explanation than can be given here.

In its most straightforward form, Stein's method requires finding a "companion" random variable W^* such that (W, W^*) is an exchangeable pair, meaning that

$$\mathbb{P}(W = w, W^* = w^*) = \mathbb{P}(W = w^*, W^* = w) \tag{21}$$

for all values of w and w^*. If we can find such a W^* and if, in addition, there is a λ between 0 and 1 such that

$$\mathbb{E}^W W^* = (1 - \lambda)W \tag{22}$$

(that is, the expected value of W^* when W is fixed at some value is $1 - \lambda$ times that value), then we may apply Stein's method.

We are interested in showing that if π is chosen uniformly from S_n, then the random variable $D = \mathrm{Des}(\pi)$ is approximately normal. This has been proven before, and in more generality; see [Ful04] for references. We will demonstrate the set-up for Stein's method—that is, finding a companion variable and showing that it satisfies Equation (21) and Equation (22). From there, applying the method would proceed as in [Ful04].

Often the companion variable in Stein's method is defined by adding a little bit of randomness to the variable we are interested in. In this case, let τ be selected uniformly from among the transpositions in S_n, independently of π. Then $\tau\pi$ is uniformly distributed over S_n, and for any $u, v \in S_n$,

$$\begin{aligned}\mathbb{P}(\pi = u, \tau\pi = v) &= \mathbb{P}(\pi = u, \tau = vu^{-1}) = \mathbb{P}(\pi = u)\mathbb{P}(\tau = vu^{-1})\\ \mathbb{P}(\pi = v, \tau\pi = u) &= \mathbb{P}(\pi = v, \tau = uv^{-1}) = \mathbb{P}(\pi = v)\mathbb{P}(\tau = uv^{-1}).\end{aligned}$$

Both right-hand sides are $(n!)^{-1}\binom{n}{2}^{-1}$ if vu^{-1} is a transposition and 0 otherwise, so $(\pi, \tau\pi)$ is an exchangeable pair.

Let $D^* := \mathrm{Des}(\tau\pi)$. Since $(\pi, \tau\pi)$ is an exchangeable pair, $(F(\pi), F(\tau\pi))$ is exchangeable for any function F. So (D, D^*) is exchangeable. For $1 \le i \le n - 1$ let

$$D_i = [\pi(i) > \pi(i+1)] \quad \text{and} \quad D_i^* = [\tau\pi(i) > \tau\pi(i+1)]$$

be Bernoulli random variables; then $D = \sum_{i=1}^{n-1} D_i$ and $D^* = \sum_{i=1}^{n-1} D_i^*$.

Fix π and i and let $a = \pi(i)$, $b = \pi(i+1)$. If $a < b$, the only ways for $\tau\pi(i)$ to be bigger than $\tau\pi(i+1)$ are if τ swaps a with something bigger than b ($n-b$ ways), if τ swaps b with something smaller than a ($a-1$ ways), or if τ swaps a with b. So

$$\mathbb{E}^{D_i=0}(D_i^* - D_i) = \mathbb{P}(D_i^* = 1 | D_i = 0) = \frac{n + \pi(i) - \pi(i+1)}{\binom{n}{2}}$$

and similarly if $a > b$,

$$\mathbb{E}^{D_i=1}(D_i^* - D_i) = -\mathbb{P}(D_i^* = 0 | D_i = 1) = -\frac{n + \pi(i+1) - \pi(i)}{\binom{n}{2}}.$$

So in general

$$\mathbb{E}^{D_i}(D_i^* - D_i) = \frac{\pi(i) - \pi(i+1)}{\binom{n}{2}} + \frac{2(1 - 2D_i)}{n-1}.$$

Summing now over i causes the $\pi(i)$ terms to telescope:

$$\mathbb{E}^{\pi}(D^* - D) = \sum_{i=1}^{n-1} \mathbb{E}^{\pi}(D_i^* - D_i) = \frac{\pi(1) - \pi(n)}{\binom{n}{2}} + 2 - \frac{4D}{n-1}$$

which allows us to apply Theorem 3:

$$\begin{aligned}\mathbb{E}^{D}(D^* - D) = \mathbb{E}^{D}\mathbb{E}^{\pi}(D^* - D) &= \frac{\mathbb{E}^{D}\pi(1) - \mathbb{E}^{D}\pi(n)}{\binom{n}{2}} + 2 - \frac{4D}{n-1} \\ &= \frac{2}{n(n-1)}((D+1) - (n-D)) + 2 - \frac{4D}{n-1} = \frac{2(n-1) - 4D}{n}.\end{aligned}$$

The mean and variance of $\mathrm{Des}(\pi)$ are $\mu := (n-1)/2$ and $\sigma^2 := (n+1)/12$ respectively, so the variables

$$W := \frac{\mathrm{Des}(\pi) - \mu}{\sigma} \quad \text{and} \quad W^* := \frac{\mathrm{Des}(\tau\pi) - \mu}{\sigma}$$

have mean 0 and variance 1. Then (W, W^*) is an exchangeable pair and

$$\mathbb{E}^{W=w}(W^* - W) = \mathbb{E}^{D=\sigma w + \mu}\left(\frac{D^* - \mu}{\sigma} - \frac{D - \mu}{\sigma}\right) = \frac{1}{\sigma}\mathbb{E}^{D=\sigma w + \mu}(D^* - D)$$

which is to say

$$\mathbb{E}^{W}(W^* - W) = \frac{2(n-1) - 4(\sigma W + \mu)}{\sigma n} = -\frac{4}{n}W.$$

So if W^* is obtained using the "random transposition" method described here, (W, W^*) will be an exchangeable pair satisfying Equation (22) with $\lambda = 4/n$. One can now proceed with Stein's method and show that W is close to being a standard normal random variable.

6 Generating Functions

It follows from Equation (13) that

$$a_n(x) := \sum_d \left\langle {n \atop d} \right\rangle x^{d+1} = (1-x)^{n+1} \sum_{j\geq 0} j^n x^j$$

and therefore that

$$\begin{aligned} A(x,z) : &= \sum_{n\geq 1} a_n(x) z^n/n! = \sum_{n\geq 1} (1-x)^{n+1} \sum_{j\geq 0} j^n x^j z^n/n! \\ &= (1-x) \sum_{j\geq 0} x^j \sum_{n\geq 1} (j(1-x)z)^n/n! = (1-x) \sum_{j\geq 0} x^j \left(e^{j(1-x)z} - 1\right) \\ &= (1-x)\left(\frac{1}{1-xe^{(1-x)z}} - \frac{1}{1-x}\right) = \frac{1-x}{1-xe^{(1-x)z}} - 1 \\ &= \frac{xe^{-(1-x)z} - x}{x - e^{-(1-x)z}}. \end{aligned}$$

There is some disagreement in the literature about what a_0 should be. We have avoided the problem by not including it in the sum.

There are various ways to define generating functions for the $\left\langle {n \atop d} \right\rangle_k$, depending on which variables are kept constant.

Theorem 4

$$\sum_{n,d,k} \left\langle {n \atop d} \right\rangle_k x^d y^k z^n/n! = \frac{1}{\theta} \int_\theta^{\theta y} \frac{dt}{x - t^{1-1/y}} \tag{23}$$

where $\theta = \exp\left\{\left(\frac{1-x}{y^{-1}-1}\right) z\right\}$.

Proof. Let $B(x,y,z)$ be the left-hand side of Equation (23). Note the sum is over all integers n, d, and k. So

$$\begin{aligned} &(y^{-1}-1)\frac{\partial B}{\partial z} + (1-x)B = \\ &\sum_{n,d,k} \left[\left\langle {n+1 \atop d} \right\rangle_{k+1} - \left\langle {n+1 \atop d} \right\rangle_k + \left\langle {n \atop d} \right\rangle_k - \left\langle {n \atop d-1} \right\rangle_k \right] x^d y^k z^n/n! \end{aligned} \tag{24}$$

Let $S(n,d,k)$ be the bracketed quantity. It is clearly 0 if $n < 0$, and if $n = 0$,

$$S(0,d,k) = \begin{cases} 1 & \text{if } d = 0, k = 0 \\ -1 & \text{if } d = 0, k = 1 \\ 0 & \text{otherwise} \end{cases}$$

so $n = 0$ contributes $1 - y$ to the sum on the right-hand side of Equation (24). If $n \geq 1$, then by Equation (10), $S(n, d, k)$ is 0 unless $k = 0$ or $k = n + 1$, in which case

$$S(n,d,0) = \left\langle {n+1 \atop d} \right\rangle_1 = \left\langle {n \atop d} \right\rangle \quad \text{and} \quad S(n,d,n+1) = -\left\langle {n+1 \atop d} \right\rangle_{n+1} = -\left\langle {n \atop d-1} \right\rangle.$$

Therefore

$$\begin{aligned}
(y^{-1}-1)\frac{\partial B}{\partial z} + (1-x)B &= 1 - y + \sum_{n\geq 1,d} \left\langle {n \atop d} \right\rangle x^d z^n/n! - \sum_{n \geq 1,d} \left\langle {n \atop d-1} \right\rangle x^d y^{n+1} z^n/n! \\
&= 1 - y + x^{-1}A(x,z) - yA(x,yz) \\
&= \left[1 + x^{-1}\frac{xe^{-(1-x)z} - x}{x - e^{-(1-x)z}}\right] - y\left[1 + \frac{xe^{-(1-x)yz} - x}{x - e^{-(1-x)yz}}\right] \\
&= (1-x)\left[\frac{ye^{-(1-x)yz}}{x - e^{-(1-x)yz}} - \frac{1}{x - e^{-(1-x)z}}\right].
\end{aligned}$$

Let $\alpha = \frac{1-x}{y^{-1}-1}$. Then θ, as defined in the theorem, is $e^{\alpha z}$. Dividing by $y^{-1}-1$ and multiplying through by θ gives

$$\theta\frac{\partial B}{\partial z} + \alpha\theta B = \alpha\theta\left[\frac{ye^{-(1-x)yz}}{x - e^{-(1-x)yz}} - \frac{1}{x - e^{-(1-x)z}}\right]$$

which is to say that

$$\frac{\partial}{\partial z}(\theta B) = \alpha\left[\frac{y\theta^y}{x - \theta^{y-1}} - \frac{\theta}{x - \theta^{1-1/y}}\right].$$

Differentiating the integral on the right-hand side of Equation (23),

$$\begin{aligned}
\frac{\partial}{\partial z}\int_\theta^{\theta^y} \frac{dt}{x - t^{1-1/y}} &= \frac{\partial \theta^y}{\partial z}\left[\frac{1}{x - (\theta^y)^{1-1/y}}\right] - \frac{\partial\theta}{\partial z}\left[\frac{1}{x - \theta^{1-1/y}}\right] \\
&= \frac{\alpha y \theta^y}{x - \theta^{y-1}} - \frac{\alpha\theta}{x - \theta^{1-1/y}} = \frac{\partial}{\partial z}(\theta B).
\end{aligned}$$

Since θB and the integral have the same derivative with respect to z, and they both vanish when $z = 0$, they are equal. $\square$

Here are three more generating functions. They can all be found by plugging in Equation (16) and switching summation signs.

$$\sum_d \left\langle {n \atop d} \right\rangle_k x^d = (1-x)^n \sum_{j \geq 0} j^{k-1}(j+1)^{n-k}x^j \tag{25}$$

$$\sum_k \left\langle {n \atop d} \right\rangle_k y^k = y\sum_{j\geq 0}(-1)^{d-j}\binom{n}{d-j}\frac{(j+1)^n - (jy)^n}{j+1-jy} \tag{26}$$

$$\sum_{d,k} \left\langle {n \atop d} \right\rangle_k x^d y^k = (1-x)^n y \sum_{j \geq 0}\frac{(j+1)^n - (jy)^n}{j+1-jy}x^j \tag{27}$$

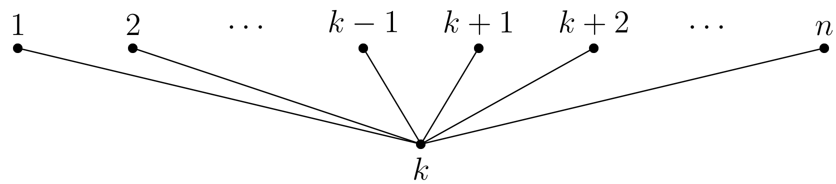
We can now prove a special case of the Neggers-Stanley conjecture. Define the descent polynomial of $A \subset S_n$ to be

$$F_A(x) = \sum_{\pi \in A} x^{\mathrm{Des}(\pi)}.$$

Let P be a poset of n elements with labels $1, 2, \ldots, n$. A linear extension of P is an ordering of $1, 2, \ldots, n$ which preserves the ordering of P; that is, a $\pi \in S_n$ which is such that if $i <_P j$ then i appears before j in the list $\pi(1), \pi(2), \ldots, \pi(n)$. If $\mathcal{L}(P)$ denotes the set of linear extentions of P, then Neggers and Stanley [Sta00, p. 311] conjectured that for any poset, every zero of $F_{\mathcal{L}(P)}$ is real.

The conjecture has been shown to be false in general [Brä04, Stear]. But we can prove it is true in a certain special case.

Theorem 5 *If $P_{n,k}$ is the poset with Hasse diagram*



then $F_{\mathcal{L}(P_{n,k})}$ has only distinct real roots.

Proof. For $u, v \geq 0$ let

$$c_{u,v} := \sum_d \left\langle {u+v+1 \atop d} \right\rangle_{u+1} x^d = \sum_{\substack{\pi \in S_{u+v+1} \\ \pi(1) = u+1}} x^{\mathrm{Des}(\pi)}.$$

Then setting $u = k-1$, $v = n-k$ yields the polynomial in question. If $v = 0$, $c_{u,v}$ counts the reversal permutation ρ, which has $(u+v+1)-1 = u$ descents. Otherwise, if $v > 0$, $c_{u,v}$ doesn't count ρ but it does count the permutation

$$u+1, u, u-1, \ldots, 1, u+v+1, u+v, \ldots, u+2$$

which has u+v-1 descents. So

$$\deg(c_{u,v}) = \begin{cases} u & \text{if } v = 0 \\ u+v-1 & \text{if } v > 0. \end{cases}$$

Similarly, if $u = 0$, $c_{u,v}$ counts the identity permutation, which has no descents. Otherwise it doesn't count the identity but it does count

$$u+1, u+2, \ldots, u+v+1, 1, 2, \ldots, u$$

which has 1 descent. So $x \nmid c_{0,v}(x)$ and if $u > 0$, $x \mid c_{u,v}(x)$ but $x^2 \nmid c_{u,v}(x)$. Now let

$$h_{u,v} := \frac{c_{u,v}}{(1-x)^{u+v+1}}.$$

$$
\begin{aligned}
& h_{0,0}(x) = (1-x)^{-1} \\
Dx\ \ & h_{0,1}(x) = (1-x)^{-2} \\
Dx\ \ & h_{0,2}(x) = (1-x)^{-3}(1+x) \\
Dx\ \ & h_{0,3}(x) = (1-x)^{-4}(1+4x+x^2) \\
xD\ \ & h_{1,3}(x) = (1-x)^{-5}(8x+14x^2+2x^3) \\
xD\ \ & h_{2,3}(x) = (1-x)^{-6}(8x+60x^2+48x^3+4x^4) \\
xD\ \ & h_{3,3}(x) = (1-x)^{-7}(8x+160x^2+384x^3+160x^4+8x^5)
\end{aligned}
$$

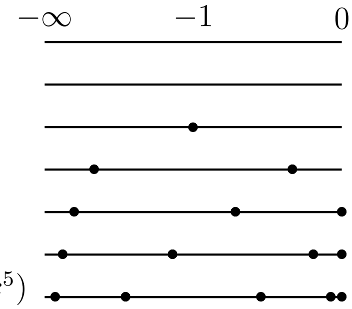


Figure 2: The construction of $h_{3,3}(x)$ as described in the proof of Theorem 5. The zeroes of each function are plotted on the right, using an inverse tangent scale. Since each function is generated from the previous one by applying either the Dx or the xD operator, Rolle's Theorem guarantees that the zeroes must interleave. By a counting argument, all the zeroes of each function must be real.

Note that $c_{u,v}(1) = \#\{\pi \in S_{u+v+1} : \pi(1) = u+1\} = (u+v)!$, so $c_{u,v}$ does not have a zero at $x = 1$. Therefore $h_{u,v}$ has exactly the same zeroes as $c_{u,v}$, plus a pole at $x = 1$. By Equation (25),

$$h_{u,v}(x) = \sum_{j \geq 0} j^u (j+1)^v x^j.$$

If D represents differentiation with respect to x, we have

$$(xD)h_{u,v}(x) = h_{u+1,v}(x) \quad \text{and} \quad (Dx)h_{u,v}(x) = h_{u,v+1}(x)$$

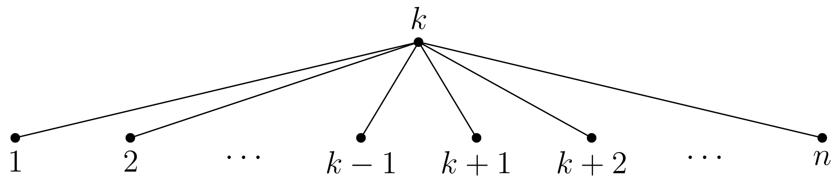
and so

$$h_{0,v}(x) = (Dx)^v h_{0,0}(x) \quad \text{and} \quad h_{u,v}(x) = (xD)^u h_{0,v}(x).$$

$h_{0,0}(x) = (1-x)^{-1}$ and $h_{0,1}(x) = (1-x)^{-2}$ both have no zeroes. Suppose $v \geq 1$ and $h_{0,v}$ has only distinct real zeroes. Since $\deg(c_{0,v}) = v-1$ and $x \nmid c_{0,v}(x)$, $xc_{0,v}(x)$ and $xh_{0,v}(x)$ have v distinct real zeroes. By Rolle's Theorem, $(Dx)h_{0,v}$ must have $v-1$ distinct zeroes interlaced between those of $xh_{0,v}(x)$. Furthermore, the denominator of $xh_{0,v}(x)$ has degree $v+1$, so $xh_{0,v}(x)$ approaches 0 as $x \to \infty$. Therefore its graph must turn back toward the x-axis somewhere to the left of its leftmost zero, at which place there must be another zero of $(Dx)h_{0,v}$. So we have found v real zeroes of $h_{0,v+1}$, and that accounts for all its zeroes.

Applying the xD operator goes similarly. Given that $h_{u,v}$ has d distinct real zeroes, by Rolle's Theorem $Dh_{u,v}(x)$ has $d-1$ interlaced zeroes. Since the numerator of $h_{u,v}$ has degree smaller than the denominator, $h_{u,v}$ must turn back toward the axis to the left of its leftmost zero, which accounts for one more zero of $Dh_{u,v}$. Finally, $(xD)h_{u,v}$ has one more zero at 0 (which is distinct from the others since $x^2 \nmid h_{u,v}$ and therefore $x \nmid Dh_{u,v}$). So we have found $d+1$ real zeroes of $h_{u+1,v}$, and that accounts for all of the zeroes. □

Corollary 6 *The same can be said for the poset*



Proof. The result of turning a poset upside-down is to reverse all its linear extensions, which changes ascents to descents and vice-versa. So if $F(x)$ is the descent polynomial of the original poset, the descent polynomial of the new poset is $x^{n-1}F(x^{-1})$. So the roots of the new polynomial are the inverses of the roots of the original. □

7 General Behavior

We can say in general how the sequence $\left\langle {n \atop d} \right\rangle_n, \left\langle {n \atop d} \right\rangle_{n-1}, \ldots, \left\langle {n \atop d} \right\rangle_1$ behaves.

The set of numbers $\left\langle {n \atop d} \right\rangle_k$, for n fixed, is very nearly unimodal if arranged appropriately.

Theorem 7 *Fix n and d. Then*

(i) *If $d = 0$,* $\quad 0 = \left\langle {n \atop d} \right\rangle_n = \cdots = \left\langle {n \atop d} \right\rangle_2 < \left\langle {n \atop d} \right\rangle_1 = 1$

(ii) *If $1 \le d \le (n-3)/2$,* $\quad \left\langle {n \atop d} \right\rangle_n < \left\langle {n \atop d} \right\rangle_{n-1} < \cdots < \left\langle {n \atop d} \right\rangle_1$

(iii) *If n is even and $d = (n-2)/2$,* $\quad \left\langle {n \atop d} \right\rangle_n < \cdots < \left\langle {n \atop d} \right\rangle_2 = \left\langle {n \atop d} \right\rangle_1$

(iv) *If n is odd and $d = (n-1)/2$,* $\quad \left\langle {n \atop d} \right\rangle_n < \cdots < \left\langle {n \atop d} \right\rangle_{(n+1)/2} > \cdots > \left\langle {n \atop d} \right\rangle_1$

(v) *If n is even and $d = n/2$,* $\quad \left\langle {n \atop d} \right\rangle_n = \left\langle {n \atop d} \right\rangle_{n-1} > \cdots > \left\langle {n \atop d} \right\rangle_1$

(vi) *If $(n+1)/2 \le d \le n-2$,* $\quad \left\langle {n \atop d} \right\rangle_n > \left\langle {n \atop d} \right\rangle_{n-1} > \cdots > \left\langle {n \atop d} \right\rangle_1$

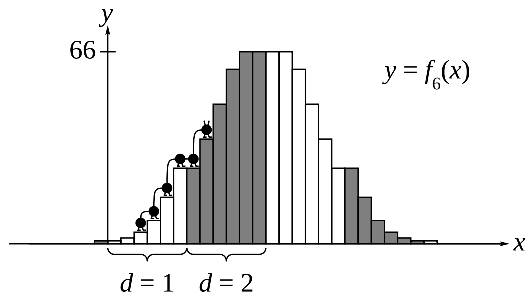
(vii) *If $d = n-1$,* $\quad 1 = \left\langle {n \atop d} \right\rangle_n > \left\langle {n \atop d} \right\rangle_{n-1} = \cdots = \left\langle {n \atop d} \right\rangle_1 = 0.$

Proof. (i) follows from the fact that the identity is the only permutation with 0 descents. (v), (vi), and (vii) follow from (iii), (ii), and (i) respectively because $\left\langle {n \atop d} \right\rangle_k = \left\langle {n \atop n-1-d} \right\rangle_{n+1-k}$.

Let $f_n(x) = \left\langle {n \atop \lfloor x/n \rfloor + 1} \right\rangle_{n\lfloor x/n \rfloor + n - x}$, which means that $f_n(nd-k) = \left\langle {n \atop d} \right\rangle_k$ if $0 \le d \le n-1$ and $1 \le k \le n$. Figure 3 shows the graphs of $f_6(x)$ and $f_7(x)$. Each monochromatic section is a sequence of the form $\left\langle {n \atop d} \right\rangle_n, \left\langle {n \atop d} \right\rangle_{n-1}, \ldots, \left\langle {n \atop d} \right\rangle_1$. Note the graphs plateau where one sequence meets the next. Since $\left\langle {n \atop d} \right\rangle_1 = \left\langle {n-1 \atop d} \right\rangle = \left\langle {n \atop d+1} \right\rangle_n$, each sequence begins where the previous one ends. The content of the theorem is that f_n is basically unimodal. That is, the sequences on the left increase, those on the right decrease, and those in the middle behave according to (iii) through (v).

The theorem is true for small n by inspection. By Equation (9),

$$\left\langle {n+1 \atop d} \right\rangle_k = \sum_{\ell=1}^{k-1} \left\langle {n \atop d-1} \right\rangle_\ell + \sum_{\ell=k}^{n} \left\langle {n \atop d} \right\rangle_\ell = \sum_{\ell=1}^{k-1} f_n(n(d-1)-\ell) + \sum_{\ell=k}^{n} f_n(nd-\ell).$$



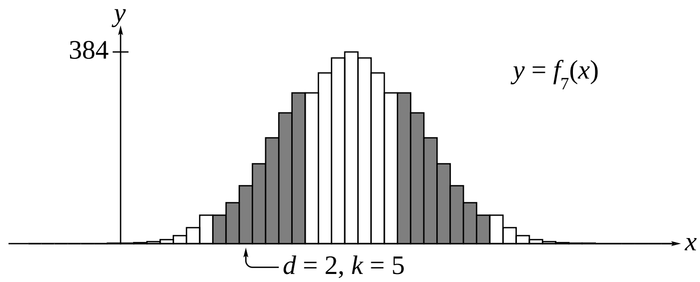


Figure 3: The graphs of $f_6(x)$ and $f_7(x)$, where $f_n(nd-k) = \left\langle {n \atop d} \right\rangle_k$, as defined in Theorem 7.

Let $i = \ell + n - k$ in the first sum and $\ell - k$ in the second and we have

$$\left\langle {n+1 \atop d} \right\rangle_k = \sum_{i=n-k+1}^{n-1} f(n(d-1) - (i-n+k)) + \sum_{i=0}^{n-k} f(nd - (i+k)) = \sum_{i=0}^{n-1} f(nd-k-i).$$

So imagine a caterpillar of length n crawling on the graph of $y = f_n(x)$, as shown in the top graph of Figure 3. If his head is at x-position $nd-k$, the equation above says that the sum of the heights of his segments (or his total potential energy) is $\left\langle {n+1 \atop d} \right\rangle_k$. If he were to take a step forward, his total energy would be $\left\langle {n+1 \atop d} \right\rangle_{k+1}$. That would be an increase in energy if the new height of his head is higher than the current height of his tail. The theorem now follows easily by induction. □

8 Behavior if $d \ll n$

If d is much less than n, and π is selected at random from those permutations of n letters which have d descents, then the distribution of $\pi(1)$ approaches a geometric distribution uniformly, in the following sense.

Theorem 8 *Fix an integer $d > 0$. Suppose π_n is chosen uniformly from those permutations* *of n letters which have d descents. Then for any $\epsilon > 0$ there is an N such that*

$$\left|\frac{\mathbb{P}(\pi_n(1) = k)}{(1-p)p^{k-1}} - 1\right| < \epsilon \tag{28}$$

for all integers n and k with $n \geq N$ and $1 \leq k \leq n$, where $p = \frac{d}{d+1}$.

Proof. For $0 \leq j \leq d$, let $P_j(n) = (-1)^{d-j}\binom{n}{d-j}$. Then by Equation (16) and Equation (13),

$$\left\langle {n \atop d} \right\rangle_k = d^{k-1}(d+1)^{n-k} \sum_{0 \leq j \leq d} P_j(n) \left(\frac{j}{d}\right)^{k-1} \left(\frac{j+1}{d+1}\right)^{n-k} \tag{29}$$

$$\left\langle {n \atop d} \right\rangle = (d+1)^n \sum_{0 \leq j \leq d} P_j(n+1) \left(\frac{j+1}{d+1}\right)^n. \tag{30}$$

Since $(1-p)p^{k-1} = d^{k-1}/(d+1)^k$, the left-hand side of Equation (28) is

$$\left|\frac{\sum_{0 \leq j \leq d} P_j(n) \left(\frac{j}{d}\right)^{k-1} \left(\frac{j+1}{d+1}\right)^{n-k}}{\sum_{0 \leq j \leq d} P_j(n+1) \left(\frac{j+1}{d+1}\right)^n} - 1\right|$$

and since $P_d(n) = \binom{n}{0} = 1$, the last term of both sums is 1. Therefore we have

$$\left|\frac{\sum_{0 \leq j < d} \left[P_j(n) \left(\frac{j}{d}\right)^{k-1} \left(\frac{j+1}{d+1}\right)^{n-k} - P_j(n+1) \left(\frac{j+1}{d+1}\right)^n\right]}{1 + \sum_{0 \leq j < d} P_j(n+1) \left(\frac{j+1}{d+1}\right)^n}\right|.$$

Since $j/d < (j+1)/(d+1)$ when $0 \leq j < d$, that's bounded above by

$$\frac{\sum_{0 \leq j < d} \left[|P_j(n)| \left(\frac{j+1}{d+1}\right)^{n-1} + |P_j(n+1)| \left(\frac{j+1}{d+1}\right)^n\right]}{\left|1 + \sum_{0 \leq j < d} P_j(n+1) \left(\frac{j+1}{d+1}\right)^n\right|}.$$

Now each term in each sum is a polynomial in n times a decaying exponential in n. So both sums go to 0 as n goes to infinity. □

Corollary 9 *The total variation distance between the distribution of $\pi_n(1)$ and the geometric distribution with parameter $p = \frac{d}{d+1}$ approaches 0 as n approaches infinity.*

9 If Both Ends Are Fixed

We might now ask about the number of permutations with d descents whose first and last positions are fixed. Let

$$\left\langle {n \atop d} \right\rangle_k^\ell := \#\left\{\pi \in S_n : \mathrm{Des}(\pi) = d,\ \pi(1) = k, \text{ and } \pi(n) = \ell\right\}.$$

Theorem 10 *Suppose* $1 \le k < k+m \le n$. *Then*

$$\left\langle {n \atop d} \right\rangle_k^{k+m} = \left\langle {n-1 \atop d} \right\rangle^m \quad \text{and} \quad \left\langle {n \atop d} \right\rangle_{k+m}^{k} = \left\langle {n-1 \atop d-1} \right\rangle_m .$$

Proof. Let $\psi \in S_n$ be the n-cycle $(n, n-1, \ldots, 2, 1)$. Then for any $\pi \in S_n$,

$$\psi\pi(i) = \begin{cases} \pi(i) - 1 & \text{if } \pi(i) > 1 \\ n & \text{if } \pi(i) = 1. \end{cases}$$

(Imagine a device like a car odometer, with a window and n wheels, on each of which are painted the numbers 1 through n. π can be represented by turning the ith wheel until $\pi(i)$ shows through the window, for all i. If one then rolls all the wheels backward a notch, $\psi\pi$ shows through the window. For this reason we will refer to the transformation $\pi \mapsto \psi\pi$ as a **rollback**.)

If $1 \le i < n$, let $D_i(\pi) = [\pi(i) > \pi(i+1)]$. The pair $\pi(i), \pi(i+1)$ has one of four types:

Type		$D_i(\pi)$	$D_i(\psi\pi)$	$D_i(\psi\pi) - D_i(\pi)$
A	$1 < \pi(i) < \pi(i+1)$	0	0	0
B	$1 < \pi(i+1) < \pi(i)$	1	1	0
C	$1 = \pi(i) < \pi(i+1)$	0	1	1
D	$1 = \pi(i+1) < \pi(i)$	1	0	-1

Most pairs are of type A or B. π will have one pair of type C unless $\pi(n) = 1$ and one pair of type D unless $\pi(1) = 1$. Therefore

$$\mathrm{Des}(\psi\pi) - \mathrm{Des}(\pi) = \sum_{i=1}^{n-1} D_i(\psi\pi) - D_i(\pi) = \begin{cases} 1 & \text{if } \pi(1) = 1 \\ -1 & \text{if } \pi(n) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} P_a^b &:= \{\pi \in S_n : \ \pi(1) = a \text{ and } \pi(n) = b\} \\ Q^b &:= \{\pi \in S_{n-1} : \pi(n) = b\} . \end{aligned}$$

Consider the following sequence of bijections:

$$P_k^{k+m} \xrightarrow{\text{rollback}} P_{k-1}^{k-1+m} \xrightarrow{\text{rollback}} \cdots \xrightarrow{\text{rollback}} P_1^{1+m} \xrightarrow{\text{rollback}} P_n^m \xrightarrow{\text{shorten}} Q^m$$

where "shortening" a permutation means removing n. (See Figure /reffig:rollback for an example.) The first $k-1$ rollbacks all preserve Des, and the final one increments Des. But the shortening decrements it again, since it removes n from the front of the permutation. Therefore the net effect, across the whole sequence, is to preserve Des. So $\left\langle {n \atop d} \right\rangle_k^{k+m} = \left\langle {n-1 \atop d} \right\rangle^m$ for all d.

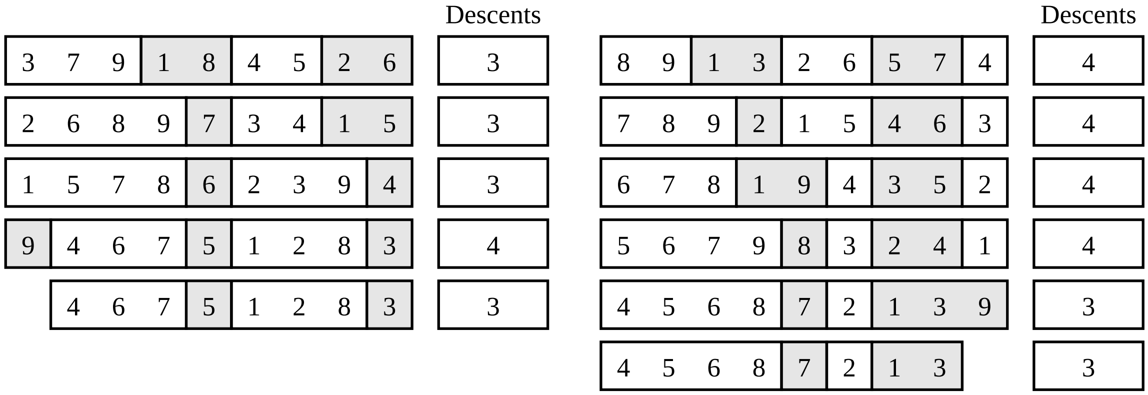


Figure 4: Examples of the actions of the bijections described in Theorem 10, for $n = 9$. Vertical lines show the positions of descents. If $\pi(1) < \pi(n)$, as at the top left, then the permutation is "rolled back" until n appears at the front, and then n is removed. In each of the rollbacks but the last, one of the internal bars moves one position to the right, to accomodate a 1 changing to an n, but the total number of descents stays the same. Only when the number in the first position changes from 1 to n do we gain a descent, but it vanishes again when we remove n in the last step. The procedure is is similar when $\pi(1) > \pi(n)$, as on the right, but the last rollback eliminates a descent, and removing n leaves the number of descents unchanged.

The second part of the theorem follows from the bijective sequence

$$P^k_{k+m} \xrightarrow{\text{rollback}} P^{k-1}_{k-1+m} \xrightarrow{\text{rollback}} \cdots \xrightarrow{\text{rollback}} P^1_{1+m} \xrightarrow{\text{rollback}} P^n_m \xrightarrow{\text{shorten}} Q_m$$

where $Q_a = \{\pi \in S_n : \pi(1) = a\}$. Here the final rollback decrements Des, and the shortening leaves it unchanged. So $\left\langle {n \atop d} \right\rangle^k_{k+m} = \left\langle {n-1 \atop d-1} \right\rangle_m$. □

Corollary 11 *If $1 \leq k, \ell \leq n$ and $P^\ell_{n,k}$ is the poset on $\{1, 2, \ldots, n\}$ defined by $k <_P a <_P \ell$ for all a other than k and ℓ, then the descent polynomial of $\mathcal{L}(P^\ell_{n,k})$ has only distinct real zeroes.*

Proof. If $\ell = k + m$, then the polynomial in question is

$$\sum_d \left\langle {n \atop d} \right\rangle^\ell_k x^d = \sum_d \left\langle {n-1 \atop d} \right\rangle^m x^d$$

which was shown to have real distinct zeroes in Corollary 6. As in that corollary, it follows immediately that turning the poset upside-down inverts the roots of the polynomial, leaving them real. □

10 Remarks

In Section 5 we noted that if π is uniformly distributed over S_n then the distribution of $D = \mathrm{Des}(\pi)$ is approximately normal. Thus the normal density function

$$\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2} \left(\frac{d-\mu}{\sigma} \right)^2 \right\}$$

with $\mu = \frac{n-1}{2}$ and $\sigma = \sqrt{\frac{n+1}{12}}$ is a good approximation for $\frac{1}{n!}\left\langle {n \atop d} \right\rangle$ when d is close to μ. However, it can be off by orders of magnitude when d is very small or very large.

Theorem 7 shows that the sequence $\left\langle {n \atop d} \right\rangle_1, \left\langle {n \atop d} \right\rangle_2, \ldots, \left\langle {n \atop d} \right\rangle_n$ is an interpolation between $\left\langle {n-1 \atop d} \right\rangle$ and $\left\langle {n-1 \atop d-1} \right\rangle$, so it seems a reasonable hypothesis that if d is close to $\frac{n-1}{2}$, then $\left\langle {n \atop d} \right\rangle_k$ is well approximated by

$$\frac{(n-1)!}{\sqrt{\frac{\pi n}{6}}} \exp\left\{ -\frac{1}{2} \left(\frac{d + \frac{n-k}{n-1} - \frac{n}{2}}{\frac{\sqrt{n}}{12}} \right)^2 \right\}.$$

Experimental evidence for $n \leq 200$ suggests that this is in fact the case. So while the distribution of $\pi(1)$ given $\mathrm{Des}(\pi)$ is by no means normal, it does seem to behave like a segment of the normal curve when d is near $\frac{n-1}{2}$.

More generally, there may be some underlying curve which the Eulerian numbers, properly normalized, can be said to approach as n grows large. It will look like a bell curve, but not be exactly normal, since the normal approximation is not very good when $d \ll n$. If so, it seems likely that the refined Eulerian numbers presented in this paper can be said to approach points on the same curve.

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