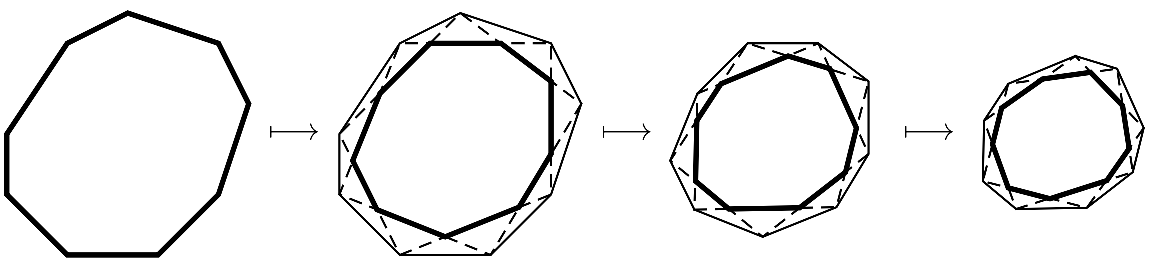
RESEARCH STATEMENT
Max Glick

My research lies in the field of algebraic combinatorics, and more specifically, cluster algebras. I am interested in better understanding the interplay between cluster algebras and discrete integrable systems. My research has focused on a particular system called the pentagram map. By identifying the pentagram recurrence as a special instance of a Y-pattern (in the setting of cluster algebra theory), I was able to obtain new results about the pentagram map, including a combinatorial formula for its iterates and many instances of the singularity confinement phenomenon. This research statement gives a brief overview of these results, together with several open problems and ideas for future projects.

The pentagram map and cluster algebras

The pentagram map, introduced by R. Schwartz [10], is an operation that produces one polygon from another by drawing its "shortest diagonals" (a precise definition will follow). Three successive applications of the pentagram map are pictured below:



Schwartz [12] noted that the pentagram map naturally extends to *twisted polygons*. A twisted n-gon is a sequence $A = (A_i)_{i\in\mathbb{Z}}$ of points in the projective plane (the vertices of A) together with a projective transformation ϕ such that $A_{i+n} = \phi(A_i)$ for all i. When ϕ is the identity, the usual notion of a (closed) polygon is recovered. Let $\mathcal{P}_n$ denote the space of twisted n-gons modulo projective equivalence. It will be convenient to also consider twisted polygons with vertices indexed by $\frac{1}{2}+\mathbb{Z}$ instead of $\mathbb{Z}$. Let $\mathcal{P}_n^*$ denote the space of projective equivalence classes of twisted n-gons indexed by $\frac{1}{2}+\mathbb{Z}$.

For $A = (A_j)$ a generic twisted polygon, let $a_{i,j}$ denote the line passing through A_i and A_j. The pentagram map, denoted T, transforms A into the twisted polygon B with vertices $B_i = a_{i-\frac{3}{2},i+\frac{1}{2}} \cap a_{i-\frac{1}{2},i+\frac{3}{2}}$. Note that if A is indexed by $\mathbb{Z}$, then B is indexed by $\frac{1}{2}+\mathbb{Z}$, and vice versa. As the pentagram map preserves projective equivalence, it induces generically defined mappings $\mathcal{P}_n \to \mathcal{P}_n^*$ and $\mathcal{P}_n^* \to \mathcal{P}_n$. The kth iterate of the pentagram map, denoted T^k, sends $\mathcal{P}_n$ to $\mathcal{P}_n$ if k is even, and to $\mathcal{P}_n^*$ if k is odd.

Schwartz demonstrated that the spaces of twisted polygons can be coordinatized so that the pentagram map becomes birational. A modification of Schwartz's coordinate system introduced in [5] allowed me to place the pentagram map in the context of cluster algebra theory. To describe these coordinates, we will need the notion of the *cross ratio* of four collinear points A, B, C, D, defined by $\chi(A,B,C,D) = \frac{(A-B)(C-D)}{(A-C)(B-D)}$; here the subtractions are interpreted as signed distances. Using projective duality, one can also take the cross ratio of four lines passing through a common point.

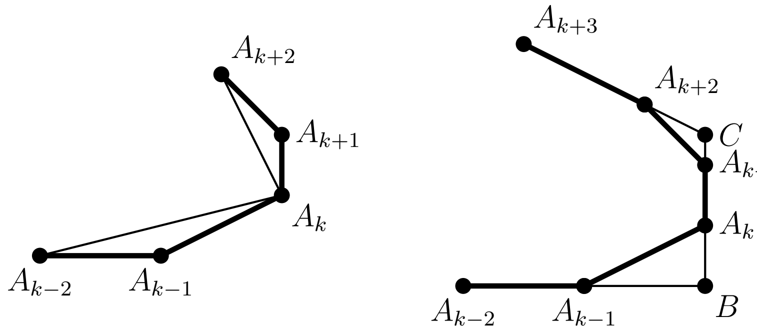


FIGURE 1. The points and lines used in the definitions of y_{2k} (left) and y_{2k+1} (right)

Definition 1. The *y-parameters* $(y_j(A))_{j\in\mathbb{Z}}$ of a twisted polygon $A \in \mathcal{P}_n \cup \mathcal{P}_n^*$ are defined by

$$y_{2k}(A) = -\left(\chi(a_{k,k-2}, a_{k,k-1}, a_{k,k+1}, a_{k,k+2})\right)^{-1},$$
$$y_{2k+1}(A) = -\chi(a_{k-2,k-1} \cap a_{k,k+1}, A_k, A_{k+1}, a_{k+2,k+3} \cap a_{k,k+1}).$$

Figure 1 illustrates these formulas.

The sequence $(y_j)_{j\in\mathbb{Z}}$ is $2n$-periodic. Unlike Schwartz's coordinates, the y-parameters $(y_1, \ldots, y_{2n})$ are not algebraically independent: they satisfy the relation $y_1 y_2 \cdots y_{2n} = 1$. However, replacing one of the y-parameters by another quantity yields a coordinate system. Moreover, the extra coordinate can be chosen to be a conserved quantity for the pentagram map, meaning that all of the interesting dynamics occurs on the level of the y_j. The following proposition explicitly describes this (birational) dynamical system.

Proposition 2 ([5, Proposition 2.3])**.** *Let $A \in \mathcal{P}_n$. Denoting $y_j = y_j(A)$, we have:*

$$y_j(T(A)) = \begin{cases} y_j^{-1}, & j \text{ even;} \\ y_j(1+y_{j-1})(1+y_{j+1})/((1+y_{j-3}^{-1})(1+y_{j+3}^{-1})), & j \text{ odd.} \end{cases} \tag{1}$$

Alternately, if $A \in \mathcal{P}_n^*$ then

$$y_j(T(A)) = \begin{cases} y_j(1+y_{j-1})(1+y_{j+1})/((1+y_{j-3}^{-1})(1+y_{j+3}^{-1})), & j \text{ even;} \\ y_j^{-1}, & j \text{ odd.} \end{cases} \tag{2}$$

The y-parameters provide a link between the pentagram map and the cluster algebras of S. Fomin and A. Zelevinsky [2]. Central to the latter theory is the dynamics of *seed mutations*, which allow one to transition from a given state (called a *seed*) in m different directions via birational maps. These transition maps are encoded in a single directed graph (a *quiver*) which is itself changed by the mutation process. Cluster algebras are closely related to *Y-patterns* (see, e.g., [3, Definition 2.9]), which are defined in terms of the dynamics of *Y-seeds*. Formulas (1)–(2) can be seen to be a special case of Y-seed mutations:

Corollary 3 ([5, Proposition 3.3])**.** *When written in the y-parameters, the pentagram map is a special case of a Y-pattern. Specifically, consider the Y-seed with quiver Q_n shown in Figure 2. Applying mutations at all black vertices, then at all white vertices, then at all black vertices, and so on, corresponds to iterating the pentagram map for the y-parameters.*

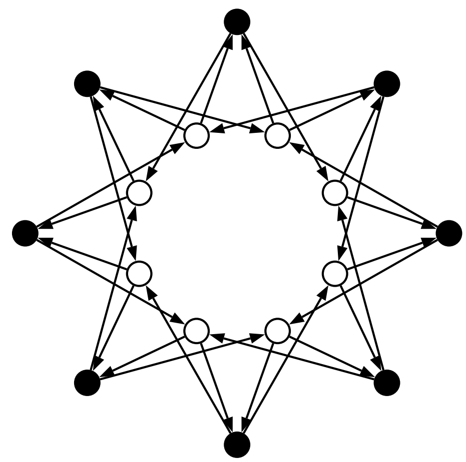


FIGURE 2. The quiver Q_n for $n = 8$

Corollary 3 makes it possible to apply general cluster theory to the study of the pentagram map. In particular, this connection can be used to obtain explicit combinatorial formulas for the iterates T^k. To state these formulas, we will need certain posets P_k introduced by N. Elkies, G. Kuperberg, M. Larsen, and J. Propp [1] in connection with height functions of domino tilings. The poset P_k can be defined as the set

$$P_k = \{(a,b,c,d) \in \mathbb{Z}^4_{\geq 0} : a+b+c+d \in \{k-2, k-1\}\}$$

partially ordered by

$$(a,b,c,d) \leq (a',b',c',d') \text{ iff } a \geq a', b \geq b', c \leq c', d \leq d'.$$

Theorem 4 ([5, Theorem 1.2])**.** *The kth iterate of the pentagram map on $\mathcal{P}_n$ is given by*

$$y_j(T^k(A)) = \begin{cases} \left(\displaystyle\prod_{i=-k}^{k} y_{j+3i}(A)\right) \dfrac{F_{j-1,k}F_{j+1,k}}{F_{j-3,k}F_{j+3,k}}, & j+k \text{ even;} \\ \left(\displaystyle\prod_{i=-k+1}^{k-1} y_{j+3i}(A)\right)^{-1} \dfrac{F_{j-3,k-1}F_{j+3,k-1}}{F_{j-1,k-1}F_{j+1,k-1}}, & j+k \text{ odd,} \end{cases} \tag{3}$$

where

$$F_{j,k} = \sum_I \prod_{(a,b,c,d)\in I} y_{3(b-a)+(d-c)+j}(A), \tag{4}$$

the sum over all order ideals I in the poset P_k.

By [3, Proposition 3.13], any y-variable of a Y-pattern is given by a formula that looks like (3), where the $F_{j,k}$ are the *F-polynomials* of the corresponding cluster algebra. The difficult part of the proof of Theorem 4 is the justification of (4), the formula for these F-polynomials.

Besides the Y-pattern defined by the quiver Q_n, one can consider various cluster algebras associated with this quiver.

Question 5. *Can the cluster variables of some cluster algebra associated to the pentagram map be given a geometric interpretation?*

The pentagram map has a natural generalization T_d that depends on an integer parameter $d > 2$. It uses d-diagonals $a_{i,i+d}$ instead of the shortest diagonals $a_{i,i+2}$. Some remarkable properties of these generalized pentagram maps were established in [11] and [13].

More recently, M. Gekhtman, M. Shapiro, S. Tabachnikov, and A. Vainshtein [4] proved that the dynamical systems defined by the maps T_d (as well as certain higher-dimensional versions thereof) are integrable. They found a connection between these systems and Y-patterns, generalizing my work reviewed above. It follows that the iterates of T_d have formulas resembling (3).

Problem 6. *Find combinatorial formulas for the F-polynomials of the cluster algebra associated to T_d for $d > 2$.*

The pentagram map and discrete integrable systems

In [12], Schwartz found a collection of quantities conserved by the pentagram map. This led to the discovery by V. Ovsienko, R. Schwartz, and S. Tabachnikov [8] that the pentagram map for twisted polygons defines a discrete integrable system. Passing to the space of closed polygons presented some difficulty, but proofs of integrability in that setting were eventually given by the same collective of authors [9] as well as by F. Soloviev [14].

In general, a singularity of a birational map f at a point x is said to be *confined* if some higher iterate f^k is nonsingular (i.e., has a well-defined value) at x. Singularity confinement was identified by B. Grammaticos, A. Ramani, and V. Papageorgiou [7] as a common feature of many discrete integrable systems. In the case of the pentagram map, Schwartz provided experimental evidence for singularity confinement several years before integrability had been proven. In [6], I proved singularity confinement of the pentagram map for a large family of singularity types, including the following:

Theorem 7 ([6, Theorem 4.2])**.** *Let $i, m \in \mathbb{Z}$ with $1 \le m < n/3 - 1$. Suppose that in a twisted polygon $A \in \mathcal{P}_n$, the points*

$$A_{i-(m+1)}, A_{i-(m-1)}, A_{i-(m-3)}, \ldots, A_{i+(m+1)}$$

are collinear. Then the map T^k has a singularity at A for $1 \le k \le m+1$. On the other hand, the polygon $T^{m+2}(A)$ is well defined for a generic A of this sort, and has $m+4$ consecutive sides passing alternately through two points on the plane.

By (1), $A \in \mathcal{P}_n$ is a singular point of T if and only if $y_{2j}(A) = -1$ for some $j \in \mathbb{Z}$. One can check that the polygons A described in Theorem 7 are precisely those satisfying $y_{2j} = -1$ for all $j \in \{i-m+1, i-m+3, \ldots, i+m-1\}$. More generally, I investigated the dependence of singularity behavior on the set $S = \{j \in \{1, 2, \ldots, n\} : y_{2j}(A) = -1\}$. For n odd, I proved [6, Corollary 4.8] that generic singularities of type S are confined unless $S = \{1, 2, \ldots, n\}$. I have a conjecture along the same lines when n is even. Still, these statements do not describe the full picture since we are yet to understand the behavior of the pentagram map at nongeneric polygons for a given S.

Problem 8. *Describe the stratification of the singular locus of T by singularity type. Find explicit descriptions for the individual strata in the language of incidence geometry.*

Let a twisted polygon A be a singular point of T but not of T^k for some $k > 1$. Then the polygon $B = T^k(A)$ is well defined even though $T(A)$ is not. It is natural to ask if there is a way to construct B from A geometrically. Under the assumptions of Theorem 7, I found a straightedge construction which bypasses the singularity and produces B [6, Section 8].

Central to this construction are objects called *decorated polygons*. A decorated polygon is a twisted polygon together with a line through each of its vertices and a point on each of its sides. The added lines and points are called *decorations*, and are required to satisfy a certain relation. Roughly, it must be possible to move the vertices locally along their respective decorating lines in such a way that the first order behavior of each side of the polygon is given by rotation through its own decorating point.

The pentagram map can be extended to a map $\tilde{T}$ on the space of decorated polygons. In fact, decorated polygons can be identified with projectivized tangent vectors in the space of polygons, and from this point of view, $\tilde{T}$ corresponds to the differential of T. I found [6, Section 6] a straightedge construction of $\tilde{T}$, which leads me to wonder if $\tilde{T}$ can be studied in a similar way to the pentagram map itself.

Question 9. *What are the "right" coordinates on the space of decorated twisted polygons? Is there a cluster-theoretic formula for $\tilde{T}$ in such coordinates? Is $\tilde{T}$ integrable in some sense?*

Another possible direction of research is to consider singularities of the generalized pentagram maps T_d described in the previous section.

Question 10. *Do the maps T_d exhibit singularity confinement?*

More generally, in any cluster algebra one can ask to what extent singularity confinement holds. Intuitively, the Laurent phenomenon for the cluster dynamics should imply confinement properties for the corresponding Y-pattern.

Question 11. *Which other Y-patterns can be shown to exhibit singularity confinement? What is the proper way to formulate this property in the setting of cluster algebra theory?*

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