Some Results on Characterizing Structures Using Infinitary Formulas

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$\mathcal{L}_{\omega_1\omega}$ is the infinitary logic which allows countably infinite conjunctions and disjunctions.

There is a hierarchy of $\mathcal{L}_{\omega_1\omega}$-formulas based on their quantifier complexity, after putting them in normal form. Formulas are classified as either $\Sigma^{\mathtt{in}}_{\alpha}$ or $\Pi^{\mathtt{in}}_{\alpha}$, for $\alpha < \omega_1$.

- A formula is $\Sigma^{\mathtt{in}}_{0}$ and $\Pi^{\mathtt{in}}_{0}$ is it is finitary quantifier-free.
- A formula is $\Sigma^{\mathtt{in}}_{\alpha}$ if it is a disjunction of formulas $(\exists \bar{y})\varphi(\bar{x}, \bar{y})$ where φ is $\Pi^{\mathtt{in}}_{\beta}$ for $\beta < \alpha$.
- A formula is $\Pi^{\mathtt{in}}_{\alpha}$ if it is a conjunction of formulas $(\forall \bar{y})\varphi(\bar{x}, \bar{y})$ where φ is $\Sigma^{\mathtt{in}}_{\beta}$ for $\beta < \alpha$.

We will also consider computable $\mathcal{L}_{\omega_1\omega}$-formulas, where the conjunctions and disjunctions are over c.e. sets.
We denote these by $\Sigma^{\mathtt{c}}_{\alpha}$ and $\Pi^{\mathtt{c}}_{\alpha}$.

Example

There is a Π^c_2 formula which describes the class of torsion groups. It consists of the group axioms together with:

$$(\forall x) \bigvee_{n \in \mathbb{N}} nx = 0.$$

Example

There is a Σ^c_1 formula which describes the dependence relation on triples x, y, z in a $\mathbb{Q}$-vector space:

$$\bigvee_{(a,b,c) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}} ax + by + cz = 0$$

Example

There is a Σ^c_3 sentence which says that a $\mathbb{Q}$-vector space has finite dimension:

$$\bigvee_{n\in\mathbb{N}} (\exists x_1, \ldots, x_n)(\forall y)\ y \in \operatorname{span}(x_1, \ldots, x_n).$$

Example

There is a Π^c_3 sentence which says that a $\mathbb{Q}$-vector space has infinite dimension:

$$\bigwedge_{n\in\mathbb{N}} (\exists x_1, \ldots, x_n) \operatorname{Indep}(x_1, \ldots, x_n).$$

Let $\mathcal{A}$ be a countable structure.

Theorem (Scott)

There is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that:

$$\mathcal{B} \text{ countable}, \mathcal{B} \models \varphi \iff \mathcal{B} \cong \mathcal{A}.$$

φ is a *Scott sentence* of $\mathcal{A}$.

Example

$(\omega, <)$ has a Π_3^c Scott sentence consisting of the Π_2^c axioms for linear orders together with:

$$\forall y_0 \bigvee_{n\in\omega} \exists y_n < \cdots < y_1 < y_0 \left[\forall z \ (z > y_0) \vee (z = y_0) \vee (z = y_1) \vee \cdots \vee (z = y_n)\right].$$

Definition (Montalbán)

$SR(\mathcal{A})$ is the least ordinal α such that $\mathcal{A}$ has a $\Pi^{\mathtt{in}}_{\alpha+1}$ Scott sentence.

Theorem (Montalbán)

Let $\mathcal{A}$ be a countable structure, and α a countable ordinal. TFAE:

- *$\mathcal{A}$ has a $\Pi^{\mathtt{in}}_{\alpha+1}$ Scott sentence.*
- *Every automorphism orbit in $\mathcal{A}$ is $\Sigma^{\mathtt{in}}_{\alpha}$-definable without parameters.*
- *$\mathcal{A}$ is uniformly (boldface) $\boldsymbol{\Delta}^0_{\alpha}$-categorical without parameters.*

Let $\mathcal{A}$ be a computable structure.

Theorem (Nadel)

$\mathcal{A}$ has Scott rank $\leq \omega_1^{CK} + 1$.

Moreover:

- $SR(\mathcal{A}) < \omega_1^{CK}$ if $\mathcal{A}$ has a computable Scott sentence.
- $SR(\mathcal{A}) = \omega_1^{CK}$ if each automorphism orbit is definable by a computable formula, but $\mathcal{A}$ does not have a computable Scott sentence.
- $SR(\mathcal{A}) = \omega_1^{CK} + 1$ if there is an automorphism orbit which is not defined by a computable formula.

There are well-known examples of structures of Scott rank $\omega_1^{CK}+1$; in particular, the Harrison linear order.

Theorem (Harrison)

There is a computable linear order of order type $\omega_1^{CK}(1+\mathbb{Q})$ which has Scott rank $\omega_1^{CK}+1$.

The original examples of computable structures of Scott rank ω_1^{CK} were built from a "homogeneous thin tree". Makkai first built a Δ_2^0 structure of Scott rank ω_1^{CK}, and Knight and Millar improved this to get a computable structure.

Theorem (Makkai, Knight-Millar)

There is a computable structure of Scott rank ω_1^{CK}.

Until recently, these were essentially all of the examples we had.

Because there are so few examples of computable structures of high Scott rank, there are many general questions about them that we don't know the answer to.

We will answer some of these questions.

Definition

Given a model $\mathcal{A}$, we define the computable infinitary theory of $\mathcal{A}$,

$$Th_\infty(\mathcal{A}) = \{\varphi \text{ a computable } \mathcal{L}_{\omega_1\omega} \text{ sentence} \mid \mathcal{A} \vDash \varphi\}.$$

The computable infinitary theory of the Makkai-Knight-Millar structure was $\aleph_0$-categorical.

Question (Millar-Sacks)

Is there a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical?

Any other models of the same theory would necessarily be non-computable and of Scott rank at least $\omega_1^{CK} + 1$.

Theorem (Millar-Sacks)

There is a structure $\mathcal{A}$ of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical.

$\mathcal{A}$ is not computable, but $\omega_1^{\mathcal{A}} = \omega_1^{CK}$. ($\mathcal{A}$ lives in a fattening of $\mathcal{L}_{\omega_1^{CK}}$.)

Freer generalized this to arbitrary admissible ordinals.

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Theorem (HT-Igusa-Knight)

There is a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical.

Definition

$\mathcal{A}$ is computably approximable if every computable infinitary sentence φ true in $\mathcal{A}$ is also true in some computable $\mathcal{B} \not\cong \mathcal{A}$ with $SR(\mathcal{B}) < \omega_1^{CK}$.

The Harrison linear order is computably approximated by the computable ordinals.

Question (Goncharov, Calvert, Knight)

Is every computable model of high Scott rank computably approximable?

Theorem (HT)

There is a computable model $\mathcal{A}$ of Scott rank $\omega_1^{CK}+1$ and a Π_2^c sentence ψ such that:

- $\mathcal{A} \models \psi$
- $\mathcal{B} \models \psi \implies SR(\mathcal{B}) = \omega_1^{CK}+1$.

The same is true for Scott rank ω_1^{CK}.

Corollary

There are computable models of Scott rank ω_1^{CK} and $\omega_1^{CK}+1$ which are not computably approximable.

I was initially interested in a different question.

Let φ be a sentence of $\mathcal{L}_{\omega_1\omega}$.

Definition

The *Scott spectrum* of φ is the set

$$\mathrm{SS}(T) = \{\alpha \in \omega_1 \mid \alpha \text{ is the Scott rank of a countable model of } T\}.$$

Question

Classify the Scott spectra.

Theorem (HT, in ZFC + PD)

The Scott spectra of $\mathcal{L}_{\omega_1\omega}$-sentences are exactly the sets of the following forms, for some $\boldsymbol{\Sigma}^1_1$ class of linear orders $\mathcal{C}$:

1. *the well-founded parts of orderings in $\mathcal{C}$,*
2. *the orderings in $\mathcal{C}$ with the non-well-founded part collapsed to a single element, or*
3. *the union of (1) and (2).*

The construction, from $\mathcal{C}$, of an $\mathcal{L}_{\omega_1\omega}$-sentence does not use PD, and:

- We can get a $\Pi^{\mathtt{in}}_2$ sentence.
- If the class $\mathcal{C}$ is lightface, then we get a Π^{c}_2 sentence.
- The Harrison linear order, with each element named by a constant, forms a Σ^1_1 class with a single member. From (1) we get $\{\omega_1^{CK}\}$ as a Scott spectrum and from (2) we get $\{\omega_1^{CK} + 1\}$.

Definition

$\mathrm{sh}(\mathcal{L}_{\omega_1,\omega})$ is the least countable ordinal α such that, for all computable $\mathcal{L}_{\omega_1\omega}$-sentences T:

$$T \text{ has a model of Scott rank } \alpha$$
$$\Downarrow$$
$$T \text{ has models of arbitrarily high Scott ranks.}$$

Question (Sacks)

What is $\mathrm{sh}(\mathcal{L}_{\omega_1,\omega})$?

Definition

$\mathrm{sh}(\mathcal{L}_{\omega_1,\omega})$ is the least countable ordinal α such that, for all computable $\mathcal{L}_{\omega_1\omega}$-sentences T:

$$T \text{ has a model of Scott rank } \alpha \\ \Downarrow \\ T \text{ has models of arbitrarily high Scott ranks.}$$

Question (Sacks)

What is $\mathrm{sh}(\mathcal{L}_{\omega_1,\omega})$?

Theorem (Sacks, Marker, HT)

$\mathrm{sh}(\mathcal{L}_{\omega_1,\omega}) = \delta^1_2$, *the least ordinal which has no* Δ^1_2 *presentation.*

Question

Classify the Scott spectra of $\mathcal{L}_{\omega_1\omega}$-sentences in ZFC.

Question

Classify the Scott spectra of computable $\mathcal{L}_{\omega_1\omega}$-sentences.

Question

Classify the Scott spectra of first-order theories.

Now we will talk about finitely-generated structures, which all have very low Scott rank.

φ is d-$\Sigma_\alpha^{\mathtt{in}}$ if it is a conjunction of a $\Sigma_\alpha^{\mathtt{in}}$ formula and a $\Pi_\alpha^{\mathtt{in}}$ formula.

Theorem (D. Miller)

Let $\mathcal{A}$ be a countable structure. If $\mathcal{A}$ has a $\Sigma_\alpha^{\mathtt{in}}$ Scott sentence, and also has a $\Pi_\alpha^{\mathtt{in}}$ Scott sentence, then $\mathcal{A}$ has a d-$\Sigma_\beta^{\mathtt{in}}$ Scott sentence for some $\beta < \alpha$.

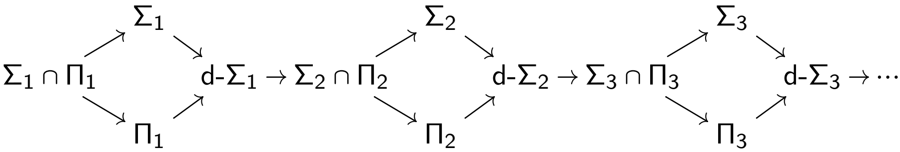
Theorem (Alvir-Knight-McCoy)

This is also true for computable sentences.

Theorem (Knight-Saraph)

Every finitely generated structure has a $\Sigma_3^{\mathtt{in}}$ *Scott sentence.*

Often there is a simpler Scott sentence.



Example

A Scott sentence for the group $\mathbb{Z}$ consists of:

- the axioms for torsion-free abelian groups,
- for any two elements, there is an element which generates both,
- there is a non-zero element with no proper divisors:

$$(\exists g \neq 0) \bigwedge_{n \in \mathbb{N}} (\forall h)[nh \neq g].$$

Example (CHKLMMMQW)

A Scott sentence for the free group $\mathbb{F}_2$ on two elements consists of:

- the group axioms,
- every finite set of elements is generated by a 2-tuple,
- there is a 2-tuple $\bar{x}$ with no non-trivial relations such that for every 2-tuple $\bar{y}$, $\bar{x}$ cannot be expressed as an "imprimitive" tuple of words in $\bar{y}$.

A pair u, v of words is primitive if whenever $\bar{x}$ is a basis for $\mathbb{F}_2$, $u(\bar{x}), v(\bar{x})$ is also a basis for $\mathbb{F}_2$.

Theorem (Knight-Saraph, CHKLMMMQW, Ho)

The following groups all have d-$\Sigma_2^{\mathtt{in}}$ Scott sentences:

- *abelian groups,*
- *free groups,*
- *nilpotent groups,*
- *polycyclic groups,*
- *lamplighter groups,*
- *Baumslag-Solitar groups $BS(1, n)$.*

Question

Does every finitely generated group have a d-$\Sigma_2^{\mathtt{in}}$ Scott sentence?

Theorem (HT-Ho, Alvir-Knight-McCoy)

Let $\mathcal{A}$ be a finitely generated structure. The following are equivalent:

- *$\mathcal{A}$ has a d-$\Sigma_2^{\mathtt{in}}$ Scott sentence.*
- *$\mathcal{A}$ does not contain a copy of itself as a proper $\Sigma_1^{\mathtt{in}}$-elementary substructure.*
- *some (every) generating tuple of $\mathcal{A}$ is defined by a $\Pi_1^{\mathtt{in}}$ formula.*

Theorem

There is a computable finitely generated group G which does not have a d-$\Sigma_2^{\mathtt{in}}$ Scott sentence.

The construction of G uses finite cancellation theory and HNN extensions.

Theorem

There is a computable finitely generated ring $\mathbb{Z}[G]$ which does not have a d-$\Sigma_2^{\mathtt{in}}$ Scott sentence.

Theorem

Every finitely generated field has a d-$\Sigma_2^{\mathtt{in}}$ Scott sentence.

Proof sketch:

Suppose that E is a proper $\Sigma_1^{\mathtt{in}}$-elementary substructure of F, with E isomorphic to F.

Then E and F have the same transcendence degree.

So F/E is an algebraic extension.

Then the atomic type of the generators of F over E is isolated, and so cannot be realized in E.

Definition

φ is a quasi-Scott sentence for a finitely generated structure $\mathcal{A}$ if φ characterizes $\mathcal{A}$ among finitely generated structures.

Each finitely generated structure has a $\Pi_3^{\mathtt{in}}$ quasi-Scott sentence.

Let p be the conjunction of the atomic type of a generating element.

Say that every element is generated by a tuple of atomic type p.

Conjecture

There is a finitely generated group with no d-$\Sigma_2^{\mathtt{in}}$ quasi-Scott sentence.

Question

Does every finitely presented group with solvable word problem have a d-$\Sigma_2^{\mathtt{in}}$ Scott sentence?

Question

Does every commutative ring have a d-$\Sigma_2^{\mathtt{in}}$ Scott sentence?

Question

Does every integral domain have a d-$\Sigma_2^{\mathtt{in}}$ Scott sentence?