Computable categoricity on a cone

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The main question / result

Setting: $\mathcal{A}$ a computable structure.

Suppose that $\mathcal{A}$ is a very "nice" structure.

OR

Consider behaviour on a cone.

How hard is it to compute isomorphisms between different copies of $\mathcal{A}$?

Main Result

Natural structures have degree of categoricity $0^{(\alpha)}$ *for some* α*.*

Degrees of categoricity

Definition

$\mathcal{A}$ is $\mathbf{d}$-computably categorical if $\mathbf{d}$ computes an isomorphism between $\mathcal{A}$ and any computable copy of $\mathcal{A}$.

Definition

$\mathcal{A}$ has degree of categoricity $\mathbf{d}$ if:

(1) $\mathcal{A}$ is $\mathbf{d}$-computably categorical and

(2) if $\mathcal{A}$ is $\mathbf{e}$-computably categorical, then $\mathbf{e} \geq \mathbf{d}$.

$\mathbf{d}$ is the least degree such that $\mathcal{A}$ is $\mathbf{d}$-computably categorical.

Example

$(\mathbb{N}, <)$ has degree of categoricity $0'$.

Which degrees are degrees of categoricity?

Theorem (Fokina, Kalimullin, Miller; Csima, Franklin, Shore)

If α is a computable ordinal then $0^{(\alpha)}$ is a degree of categoricity.

If α is a computable successor ordinal and $\mathbf{d}$ is d.c.e. in and above $0^{(\alpha)}$, then $\mathbf{d}$ is a degree of categoricity.

Theorem (Anderson, Csima)

(1) *There is a Σ_2^0 degree $\mathbf{d}$ which is not a degree of categoricity.*

(2) *Every non-computable hyperimmune-free degree is not a degree of categoricity.*

Question

Which degrees are a degree of categoricity?

Strong degrees of categoricity

Definition

d is a strong degree of categoricity for $\mathcal{A}$ if

(1) $\mathcal{A}$ is **d**-computably categorical and

(2) there are computable copies $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$ such every isomorphism $f : \mathcal{A}_1 \to \mathcal{A}_2$ computes **d**.

Every known example of a degree of categoricity is a strong degree of categoricity.

Question (Fokina, Kalimullin, Miller)

Is every degree of categoricity a strong degree of categoricity?

Natural structures

We will answer these questions for "natural structures."

A "natural structure" is a structure that one would expect to encounter in normal mathematical practice, such as $(\omega, <)$, $\mathbb{Q}$, a vector space, or an algebraically closed field.

Arguments involving natural structures tend to relativize.

Relative notions of categoricity

Definition

$\mathcal{A}$ is $\mathbf{d}$-computably categorical relative to $\mathbf{c}$ if $\mathbf{d}$ computes an isomorphism between $\mathcal{A}$ and any $\mathbf{c}$-computable copy of $\mathcal{A}$.

Definition

$\mathcal{A}$ has degree of categoricity $\mathbf{d}$ relative to $\mathbf{c}$ if:

1. $\mathbf{d} \geq \mathbf{c}$,
2. $\mathcal{A}$ is $\mathbf{d}$-computably categorical relative to $\mathbf{c}$ and
3. if $\mathcal{A}$ is $\mathbf{e}$-computably categorical relative to $\mathbf{c}$, then $\mathbf{e} \geq \mathbf{d}$.

$\mathbf{d}$ is the least degree above $\mathbf{c}$ such that $\mathcal{A}$ is $\mathbf{d}$-computably categorical relative to $\mathbf{c}$.

Cones and Martin measure

Definition

The cone of Turing degrees above $\mathbf{c}$ is the set

$$C_{\mathbf{c}} = \{\mathbf{d} : \mathbf{d} \geq \mathbf{c}\}.$$

Theorem (Martin, assuming AD)

Every set of Turing degrees either contains a cone, or is disjoint from a cone.

Think of sets containing a cone as "large" or "measure one" and sets not containing a cone as "small" or "measure zero." Note that the intersection of countably many cones contains another cone.

Relativizing to a cone

Suppose that P is a property that relativizes.

Then property P holds on a cone if it holds relative to all degrees $\mathbf{d}$ on a cone.

A natural structure has some property P if and only if it has property P on a cone.

So we can study natural structures by studying all structure relative to a cone.

The main theorem

Let $\mathcal{A}$ be a countable structure.

Main Result

Relative to a cone:

$\mathcal{A}$ *has strong degree of categoricity* $0^{(\alpha)}$ *for some ordinal* α.

More precisely:

Main Result (precisely stated)

There is an ordinal α *such that for all degrees* $\mathbf{c}$ *on a cone,* $\mathcal{A}$ *has strong degree of categoricity* $\mathbf{c}^{(\alpha)}$ *relative to* $\mathbf{c}$.

α is the Scott rank of $\mathcal{A}$.

Isomorphisms of c.e. degree

On a cone:

Theorem

Suppose that $\mathcal{A}$ is Δ^0_2-categorical. Then for every copy $\mathcal{B}$ of $\mathcal{A}$, there is a degree $\mathbf{d}$ c.e. in and above $\mathcal{B}$ such that:

(1) *every isomorphism between $\mathcal{A}$ and $\mathcal{B}$ computes $\mathbf{d}$, and*

(2) *$\mathbf{d}$ computes some isomorphism between $\mathcal{A}$ and $\mathcal{B}$.*

Corollary

Suppose that $\mathcal{A}$ is Δ^0_2-categorical and almost rigid. Then for every copy $\mathcal{B}$ of $\mathcal{A}$, every isomorphism between $\mathcal{A}$ and $\mathcal{B}$ is of c.e. degree in and above $\mathcal{B}$.