Computable Functors and Effective Interpretability

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The main theorem (stated roughly)

All structures are countable with domain ω.

Throughout, $\mathcal{A}$ and $\mathcal{B}$ will be structures.

Theorem

There is a correspondence between "effective interpretations" and "computable functors".

Example

Let $\mathcal{A}$ be the equivalence structure with one equivalence class of size n for each n.

Let $\mathcal{B}$ be the graph which consists of a cycle of size n for each n.

$\mathcal{A}$ is effectively interpretable in $\mathcal{B}$ (in fact, they are bi-interpretable).

Motivation

Computability	Syntactic
Muchnik reducibility	
Medvedev reducibility	
Computable functor	Σ-reducibility/effective interpretations

Relations on $\mathcal{A}^{<\omega}$

A *relation* on $\mathcal{A}$ is a subset of $\mathcal{A}^{<\omega}$ (not $\mathcal{A}^n$ for some n).

For example this allows us to code subsets of $\mathcal{A}^{<\omega} \times \omega$ as subsets of $\mathcal{A}^{<\omega}$ in an effective way using the length of tuples.

Many results which were originally proven for subsets of $\mathcal{A}^n$ still hold for subsets of $\mathcal{A}^{<\omega}$.

R.i.c.e. relations

Let R be a relation on $\mathcal{A}^{<\omega}$.

Definition

R is *uniformly relatively intrinsically computably enumerable (u.r.i.c.e.)* if there is a c.e. operator W such that for every copy $(\mathcal{B}, R^{\mathcal{B}})$ of $(\mathcal{A}, R)$, $R^{\mathcal{B}} = W^{D(\mathcal{B})}$.

R is *uniformly relatively intrinsically computable (u.r.i. computable)* if there is a computable operator Ψ such that for every copy $(\mathcal{B}, R^{\mathcal{B}})$ of $(\mathcal{A}, R)$, $R^{\mathcal{B}} = \Psi^{D(\mathcal{B})}$.

Recall:

Theorem (Ash-Knight-Manasse-Slaman,Chisholm)

R is u.r.i.c.e. if and only if it is definable by a Σ_1^c formula without parameters.

Effective interpretations

Let $\mathcal{A} = (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \ldots)$ where $P_i^{\mathcal{A}} \subseteq A^{a(i)}$.

Definition

$\mathcal{A}$ is *effectively interpretable* in $\mathcal{B}$ if there exist a u.r.i. computable sequence of relations $(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}, \sim, R_0, R_1, \ldots)$ such that

(1) $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \subseteq \mathcal{B}^{<\omega}$,

(2) $\sim$ is an equivalence relation on $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$,

(3) $R_i \subseteq (B^{<\omega})^{a(i)}$ is closed under $\sim$ within $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$,

and a function $f_{\mathcal{A}}^{\mathcal{B}} : \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$ which induces an isomorphism:

$$(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}/\sim; R_0/\sim, R_1/\sim, \ldots) \cong (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \ldots).$$

This is equivalent to Σ-reducibility without parameters.

Computable functors

Definition

$\mathrm{Iso}(\mathcal{A})$ is the category of copies of $\mathcal{A}$ with domain ω. The morphisms are isomorphisms between copies of $\mathcal{A}$.

Recall: a functor F from $\mathrm{Iso}(\mathcal{A})$ to $\mathrm{Iso}(\mathcal{B})$

(1) assigns to each copy $\widehat{\mathcal{A}}$ in $\mathrm{Iso}(A)$ a structure $F(\widehat{\mathcal{A}})$ in $\mathrm{Iso}(\mathcal{B})$,

(2) assigns to each isomorphism $f\colon \widehat{\mathcal{A}} \to \widetilde{\mathcal{A}}$ in $\mathrm{Iso}(\mathcal{A})$ an isomorphism $F(f)\colon F(\widehat{\mathcal{A}}) \to F(\widetilde{\mathcal{A}})$ in $\mathrm{Iso}(\mathcal{B})$.

Definition

F is *computable* if there are computable operators Φ and Φ_* such that

(1) for every $\widehat{\mathcal{A}} \in \mathrm{Iso}(\mathcal{A})$, $\Phi^{D(\widehat{\mathcal{A}})}$ is the atomic diagram of $F(\mathcal{A})$,

(2) for every isomorphism $f : \widehat{\mathcal{A}} \to \widetilde{\mathcal{A}}$, $F(f) = \Phi_*^{D(\widehat{\mathcal{A}}) \oplus f \oplus D(\widetilde{\mathcal{A}})}$.

The main theorem

Theorem

$\mathcal{A}$ is effectively interpretable in $\mathcal{B}$

$$\Updownarrow$$

there is a computable functor F from $\mathcal{B}$ to $\mathcal{A}$.

Question

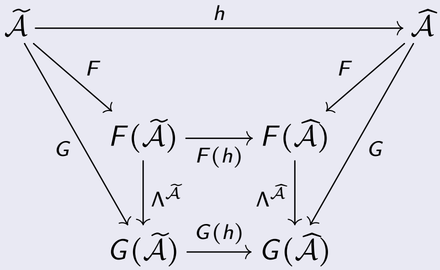
If $\mathcal{A}$ is a computable structure, is this vacuous?

Effective isomorphisms of functors

Let $F, G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ be computable functors.

Definition

F is *effectively isomorphic* to G if there is a computable Turing functional Λ such that for any $\widetilde{\mathcal{B}} \in \mathrm{Iso}(\mathcal{B})$, $\Lambda^{\widetilde{\mathcal{B}}}$ is an isomorphism from $F(\widetilde{\mathcal{B}})$ to $G(\widetilde{\mathcal{B}})$, and the following diagram commutes:



A finer analysis

Let $F\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ be a computable functor. Using the main theorem, we get an interpretation $\mathcal{I}$ of $\mathcal{A}$ in $\mathcal{B}$. Again using the main theorem, we get a functor $F_{\mathcal{I}}$ from this interpretation.

Proposition

These two functors are effectively isomorphic.

Example

Let $\mathcal{A} = \mathcal{B} = (\omega, 0, +)$. Consider the functors:

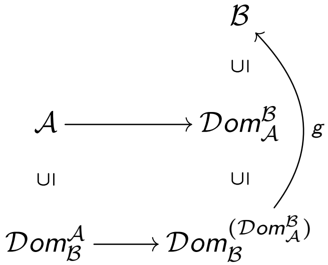
$$\begin{aligned} F &:= \text{identity functor} \\ G &:= \text{constant functor giving the standard presentation of } \omega \end{aligned}$$

These are not effectively isomorphic, and the interpretations we get are faithful to the functor.

Bi-interpretations

Definition

$\mathcal{A}$ and $\mathcal{B}$ are *effectively bi-interpretable* if there are effective interpretations of each in the other, and u.r.i. computable isomorphisms $\mathcal{D}om_{\mathcal{A}}^{(\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}})} \to \mathcal{A}$ and $\mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})} \to \mathcal{B}$.



Computable bi-transformations

Definition

$\mathcal{A}$ and $\mathcal{B}$ are *computably bi-transformable* if there are computable functors $F\colon \mathrm{Iso}(\mathcal{A}) \to \mathrm{Iso}(\mathcal{B})$ and $G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ such that both $F \circ G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{B})$ and $G \circ F\colon \mathrm{Iso}(\mathcal{A}) \to \mathrm{Iso}(\mathcal{A})$ are effectively isomorphic to the identity functor.

So if $\widehat{\mathcal{B}}$ is a copy of $\mathcal{B}$, then $F(G(\widehat{\mathcal{B}})) \cong \widehat{\mathcal{B}}$ and the isomorphism can be computed uniformly in $\widehat{\mathcal{B}}$.

Theorem

$\mathcal{A}$ and $\mathcal{B}$ are effectively bi-interpretable

$$\Updownarrow$$

$\mathcal{A}$ and $\mathcal{B}$ are computably bi-transformable.

Classes of structures

Let $\mathfrak{C}$ and $\mathfrak{D}$ be classes of structures.

Definition

$\mathfrak{C}$ is *uniformly transformally reducible* to $\mathfrak{D}$ if there is a subclass $\mathfrak{D}'$ of $\mathfrak{D}$ and computable functors $F\colon \mathfrak{C} \to \mathfrak{D}'$, $G\colon \mathfrak{D}' \to \mathfrak{C}$ such that $F \circ G$ and $G \circ F$ are effectively isomorphic to the identity functor.

Definition

$\mathfrak{C}$ is *reducible via effective bi-interpretability* to $\mathfrak{D}$ if for every $\mathcal{C} \in \mathfrak{C}$ there is a $\mathcal{D} \in \mathfrak{D}$ such that $\mathcal{C}$ and $\mathcal{D}$ are effectively bi-interpretable and the formulas involved do not depend on the choice of $\mathcal{C}$ or $\mathcal{D}$.

Theorem

$\mathfrak{C}$ is reducible via effective bi-interpretability to $\mathfrak{D}$

$$\Updownarrow$$

$\mathfrak{C}$ is uniformly transformally reducible to $\mathfrak{D}$.

Examples

Theorem (Hirschfeldt, Khoussainov, Shore, Slinko)

Every class is reducible via effective bi-interpretability to each of the following classes:

1. *undirected graphs,*
2. *partial orderings, and*
3. *lattices,*

and, after naming finitely many constants,

1. *integral domains,*
2. *commutative semigroups, and*
3. *2-step nilpotent groups.*

Theorem (Miller, Park, Poonen, Schoutens, Shlapentokh)

We can add fields of characteristic zero to the first list above.

Examples of interpretations above a jump

Theorem (Marker, Miller)

There is a computable functor from graphs to differentially closed fields (and an inverse functor, defined only on some differentially closed fields, which is $0'$*-computable).*

Theorem (Ocasio)

There is a computable functor from linear orders to real closed fields (and an inverse functor, defined only on some real closed fields, which is $0'$*-computable).*