GETTING STARTED WITH METRICS

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LOG(M) FALL 2018: POLYHEDRAL FINSLER METRICS

Welcome to Log(M)! Our goal this semester is to study polyhedral Finsler metrics and their geodesics, and write programs to draw pictures for us. There are a lot of pieces to this story, and over the first few weeks we will work together to ensure everyone is comfortable with the whole picture.

Note: These worksheets have been compiled into a single document for future reference. Feel free to contact Mark Greenfield at markjg@umich.edu with questions and comments. I would also like to acknowledge my Pat Boland and Francesca Gandini for mentoring with me and for making some suggestions on several parts of these worksheets, as well as my wonderful Log(M) team Adam Azlan, Melissa George, Yinlan Shao, and Haidan Tang who participated in this project.

1. Metric spaces

Definition 1.1. Let X be a set (of points in some unknown space, perhaps). A *metric* on X is a function $d : X \times X \to \mathbb{R}$ such that:

(1) It is *positive-definite*, that is, $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$
(2) It satisfies the *triangle inequality*: $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.
(3) It is *symmetric*: $d(x, y) = d(y, x)$ for all $x, y \in X$.

The pair (X, d) is called a *metric space*.

In this project, we will study a special class of metrics on $\mathbb{R}^n$, mostly for $n = 2$. They will sometimes be *asymmetric metrics*, where we allow condition 3 in the definition of metric to fail, so the distance from A to B might not be equal to the distance from B to A. You are already very familiar with some metric spaces!

The function $d(a, b) = |b - a|$ is a metric on $\mathbb{R}^1$. It is a special case of the Euclidean metric, with

$$d(v, w) = \sqrt{(v_n - w_n)^2 + \cdots + (v_1 - w_1)^2}$$

defining the distance between vectors in $\mathbb{R}^n$.

If some of these concepts seem new, it would be a good exercise to **prove that $\mathbb{R}^1$ and $\mathbb{R}^2$ are metric spaces with their respective Euclidean metrics**.

On any set of points, you can always define many different kinds of metrics with many different properties. Even if you have the same set of points, the metric is something defined afterwards. A set of points without a metric is just that: a set of points. Consider the *discrete metric* on $\mathbb{R}^1$:

$$d_{discr}(a, b) = \begin{cases} 1 & a \neq b \\ 0 & a = b \end{cases}$$

If you haven't seen this before, **prove this is a metric**. You might also observe that this defines a metric on any set!

Let's put a different metric on $\mathbb{R}^2$. This metric is the subject of the book *Taxicab Geometry*, by Krause. Define a new metric d_2 by:

$$d_2(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

This is called the L^1 *metric*. **Prove this is a metric on** $\mathbb{R}^2$. Can you define a similar metric on $\mathbb{R}^n$ for other n? We will end this section with one more metric on $\mathbb{R}^2$. We will call it d_∞.

$$d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

Prove this is a metric.

2. Unit balls defining metrics

Definition 2.1. Given a metric space (X, d) (even if it is asymmetric) and a point $p \in X$, define the *unit ball about* p to be the following set of points:

$$B(p, 1) = \{q \in X : d(p, q) = 1\}.$$

The order ($d(p, q)$ rather than $d(q, p)$) matters if d is asymmetric. The unit ball about a point is the set of points which are exactly distance 1 away from that point. **Draw the unit ball about the origin for each of the metrics in Section 1**. We are also often interested in radius r balls $B(p, r)$ for non-negative real numbers $r \geq 0$.

It turns out that there is a way to define (possibly asymmetric) metrics on $\mathbb{R}^n$ just by giving the unit ball. The remainder of this worksheet will focus on the case of $n = 1$. Higher dimensions will be much more complicated (and hence, worth a Log(M) project)! The earlier practice questions are standard exercises in math classes. Everything from here on out will be new and more open-ended! Your first new tasks will be to prove the following definition works, and try to understand every possibility.

Definition 2.2. A *polyhedral Finsler metric* on $\mathbb{R}^1$ is a (possibly asymmetric) metric defined in terms of a unit ball (interval) about zero. This is specified by giving a closed interval $[a, b]$ with $a < 0 < b$. The metric d is defined as:

$$d(0, y) = \lambda, \text{ such that } \lambda \geq 0 \text{ and either } \lambda b = y \text{ or } \lambda a = y$$

Distances $d(x, y)$ with both elements nonzero are determined by translating the interval $[a, b]$ to $[a + x, b + x]$ and using an analogous formula to the above.

Exercises on 1-dimensional polyhedral Finsler metrics

(1) Is the number λ even well-defined? (That is, given $[a, b]$ is there a unique one for each x?)
(2) Find a formula for distances $d(x, y)$ that only uses the definition $d(0, y')$ so you can write it more concisely.
(3) Are these symmetric or asymmetric metrics? Can you find a way to determine whether they are symmetric?
(4) Prove that these define (asymmetric or symmetric, depending on your previous answer) metrics on $\mathbb{R}^1$.
(5) What is the unit ball for a polyhedral Finsler metric on $\mathbb{R}^1$?
(6) What do the radius r balls look like?
(7) How do these compare to the Euclidean metric on $\mathbb{R}^1$?
(8) What happens if one of the endpoints of the interval is 0? What if 0 is outside the interval?
(9) Can you come up with your own questions and answer them?

POLYHEDRAL FINSLER METRICS ON $\mathbb{R}^2$

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In this worksheet, we will define and start to study the main objects of this project: Polyhedral Finsler metrics (PFMs) on the Euclidean plane. Here, we will only discuss how to define, use, and think about the actual distance function. This will not (exactly) be enough to calculate exact arc lengths of curves, and it will not be enough to more completely study geodesics (length-minimizing curves). However, we will definitely have enough here to start to ask (and maybe answer!) some interesting questions. Some of these may even be original questions.

1. The L^1 metric, and some additional metric properties

Recall from the previous worksheet, we introduced the L^1 *metric* on $\mathbb{R}^2$, which is also explored in detail in Krause's book. The distance function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by:

$$d_{L^1}(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are points in $\mathbb{R}^2$ written in their usual Euclidean coordinates. This will turn out to be a (symmetric) polyhedral Finsler metric.

Exercise 1.1. Draw the unit sphere about the origin for d_{L^1}. Draw a few different-sized spheres around other points, too. How are they related?

Exercise 1.2. How are these spheres different from unit (and other-sized) spheres about the origin (and other points) for the Euclidean metric? How are they similar? (This one is very open-ended - we don't really have a specific answer in mind!)

Here is some useful notation. Let $K \subset \mathbb{R}^2$ be a set of points in the plane, $r \in \mathbb{R}$ is a real number, and $v \in \mathbb{R}^2$ is a vector. Define the following collections:

$$rK := \{rw : w \in K\}, \quad K + v := \{w + v : w \in K\}$$

In the above definitions, think of w and v as vectors in $\mathbb{R}^2$ which can be added together and scalar multipled by r.

Exercise 1.3. Pick some fun subsets K of $\mathbb{R}^2$ and some fun examples of $r \in \mathbb{R}$ and $v \in \mathbb{R}^2$ (including nonpositive values of r) and draw rK, $K + v$, and $rK + v$ for a few different selections. How are they related to K?

Next, let's see what happens when we transform the unit sphere of a metric using the above operations.

Exercise 1.4. Let $K = B((0, 0), 1)$ be the unit sphere about the origin for d_{L^1}. For different choices of $r \in \mathbb{R}$ and $v \in \mathbb{R}^2$, what does $rK + v$ represent? Can you transform $B((0, 0), 1)$ into something else? Compare to some of the earlier exercises on this worksheet. Try this for some of the other metrics on the last worksheet.

Note that we are very crucially using here the fact that $\mathbb{R}^2$ is a vector space. For a general metric space, it might not make sense to add points together or multiply by scalars. Further, even if we could, the metric d_{L^1} looks the same everywhere. What do we mean by this?

Definition 1.5. Let V be a real vector space and d a metric on V. We say d is *translation-invariant* if for all $x, y, z \in V$, we have $d(x, y) = d(x + z, y + z)$. The metric is *self-similar* if for all $r \in \mathbb{R}$, we have $d(rx, ry) = |r| \cdot d(x, y)$.

Exercise 1.6. Is the Euclidean metric on $\mathbb{R}^2$ translation invariant? Is it self-similar? What about the L^1 metric?

Exercise 1.7. Define $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $f(x, y) = |\log(x/y)|$, where $\mathbb{R}^+$ is the set of positive real numbers. Is this a metric? Is it translation invariant? Is it self-similar?

Exercise 1.8. Can you rewrite the definitions of self-similarity and translation-invariance of a metric using only the spheres $B(p, r)$ for $p \in \mathbb{R}^2$ and $r \in \mathbb{R}^+$?

We will now use some of these ideas in reverse to build certain kinds of metrics.

2. Polyhedral Finsler metrics

So far, we have always started with a distance formula and analyzed the metric from there. Let's instead start with some special subset of $\mathbb{R}^2$, and declare it to be the unit ball for a self-similar, translation-invariant (possibly asymmetric) metric. We will give a definition of polyhedral Finsler metrics, and then we can spend some time unpacking this definition.

Definition 2.1. A *polyhedral Finsler metric* on $\mathbb{R}^2$ is a possibly asymmetric metric whose distance function is determined by a convex polygon $P \subseteq \mathbb{R}^2$ which is defined to consist of all points at a unit (forward) distance from the origin. The (forward) distance $d(p, q)$ between two points $p, q \in \mathbb{R}^n$ is defined by

$$d(p, q) = \lambda \text{ such that } q - p \in \lambda P, \text{ with } \lambda \geq 0$$

Let's study the various parts of this definition in some detail. Remember that these metrics may not be symmetric, so in your proofs, you must be careful not to mix up $d(x, y)$ and $d(y, x)$.

Exercise 2.2. Compare this definition to the case of PFMs on $\mathbb{R}^1$ discussed in the previous worksheet.

Exercise 2.3. Draw a few convex polygons in $\mathbb{R}^2$ around the origin. Plot some points in the plane and compute the distances from the origin using the formula for the polyhedral Finsler metric associated to those polygons.

Exercise 2.4. Is d_{L^1} a polyhedral Finsler metric? What about the Euclidean metric? The discrete metric? The d_∞ metric from the last worksheet?

Exercise 2.5. What is the role of P for such a metric?

Exercise 2.6. Is this even a function?? In other words, does each pair of points p, q have one and only one output λ?

Exercise 2.7. Are these positive-definite? That is, do we have $d(p, q) \geq 0$ with equality if and only if $p = q$?

Exercise 2.8. Are these self-similar and translation invariant?

Exercise 2.9. Do these satisfy the triangle inequality? **Note:** this one is quite difficult in general!

Exercise 2.10. What happens if we remove the requirement of convex? Hint: this is related to the triangle inequality.

Here is another way to define polyhedral Finsler metrics.

Definition 2.11. Let P be a convex polygon in $\mathbb{R}^2$ with the origin in the interior. The *polyhedral Finsler metric* associated to P is the unique (possibly asymmetric) metric d satisfying the following properties:

(1) P is the (forward) unit sphere from the origin
(2) d is self-similar
(3) d is translation-invariant

Exercise 2.12. Which of the previous exercises become easier or harder when using this definition instead of the first definition?

Exercise 2.13. Prove the two definitions are equivalent. Which of the two definitions do you prefer? Why? (This is purely subjective!)

Now that we have a good idea of the definition of PFMs and a few examples, we can begin to ask some more general questions. We might want to understand which PFMs have certain properties, for example. Try to come up with some of your own! Here are a few possible questions we might ask. Sometimes simple questions can lead to complicated answers!

Exercise 2.14. Which PFMs are symmetric?

Exercise 2.15. Define PFMs using polygons P and rP for some $r \in \mathbb{R}^+$. How are they related? What if $r \leq 0$?

Exercise 2.16. Define PFMs using polygons P and $P + v$ for some $v \in \mathbb{R}^2$ small enough that the origin is in both. How are they related?

Exercise 2.17. Come up with some more questions and try to answer them! There is no such thing as too easy of a question.

FINSLER METRICS ON $\mathbb{R}^2$

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Here, we will find out what "Finsler" means in Polyhedral Finsler Metric. Until now, we have only been considering distances between points and the associated structures (unit balls, etc.) one obtains from that. However, there is another extremely important notion in geometry which we will focus on for much of the semester: lengths of curves, and the related topic of curves of shortest length between two points! In Euclidean space, the shortest curve is always a straight line, but of course that depends heavily on the metric.

1. Euclidean lengths of curves

Recall from Calculus 2 and 3 that if we have a parametrized differentiable curve $\gamma : [0, 1] \to \mathbb{R}^2$ given by $\gamma(t) = (x(t), y(t))$, we can compute its length by the following formula:

$$\ell_{\mathbb{R}^2}(\gamma) = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}dt. \tag{1.1}$$

Exercise 1.2. Review the basics of parametrized curves in $\mathbb{R}^2$ (e.g. try Googling or look in some old textbooks or go to Math Lab or ask us). Draw a few and calculate their arc length! Try to compare different parametrizations for the same curve (e.g. $x(t) = t^2,\ y(t) = t^2$ parametrizes the same curve as $x(t) = t,\ y(t) = t$, both giving the line $y = x$ for $x \geq 0$).

Here is a challenge problem!

Exercise 1.3. Let $p = (x_0, y_0)$ and $q = (x_1, y_1)$ be two points in $\mathbb{R}^2$. Prove that the shortest differentiable curve between them is the (unique) line between them.

Hints: Using an isometry (rotation and translation), we may assume that $p = (0, 0)$ and $q = (x_1, 0)$. Try some integral comparison tests and see what kinds of conditions you might be able to impose to ensure a curve has shortest length.

Remark 1.4. Note that the parametrization may not be linear, but you can still get a line. You do not need to prove that the line is unique - this is basically just one of Euclid's postulates. Please email us if you want more hints! It is tricky but we do not want you to get hung up on this for too long if it does not make sense. There are a few ways to do Exercise 1.3. If you know the values of the metric (i.e. you know that $d(a, b) = c$), then you know that a shortest curve between points a and b has length c, and any curve with length c between a and b must be a curve of shortest length. Try to solve Exercise 1.3 using this information. Try again without!

2. Rectifiable curves in metric spaces*

This section is optional and does not include any exercises important for this project. I am including it since it is an interesting notion which is closely related to the topic at hand. While in this project we will mostly focus on (piecewise) differentiable curves in $\mathbb{R}^2$ or $\mathbb{R}^3$ whose length can be measured by formulas similar to Equation 1.1, it seemed relevant to introduce a notion which works in much more generality.

Let (X, d) be a metric space, and let $\gamma : [0, 1] \to X$ be a continuous function. This is a parametrized curve in X.

Remark 2.1. If you are not comfortable with continuous functions to arbitrary metric spaces, don't worry! Just pretend it is $\mathbb{R}^n$ with your usual notions of continuity. Even though the $\epsilon{-}\delta$ definition of continuity from 351/451 uses the Euclidean metric, it will turn out that this definition of continuity is fine even if we use different metrics on $\mathbb{R}^n$ for the rest of this discussion.

Let Π be the set of finite ordered subsets of $[0, 1]$, that is, elements of Π are of the form $(p_0, p_1, \ldots, p_{n-1}, p_n)$, where $p_i < p_{i+1}$, $0 \le p_0$, and $p_n \le 1$. For an element $\pi \in \Pi$, define:

$$\mathcal{L}_\pi(\gamma) = \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)).$$

This is an approximation of the length of γ by putting a bunch of points on it (the $\gamma(t_i)$'s) and looking at the actual distance (even if γ goes all over the place in between). Define the length of γ by:

$$\mathcal{L}(\gamma) = \sup_{\pi \in \Pi} \mathcal{L}_\pi(\gamma).$$

If this supremum is finite, we say γ is a *rectifiable curve* in X.

Exercise 2.2. This is kind of a fake exercise. Draw some pictures of this situation (just rough sketches of curves in space) and think about what happens. Why do we take the supremum? Why does this give us an approximation of length? What happens if γ is discontinuous? Compare this to the definition of lower sums for the Riemann integral (a topic from 351/451).

Utilizing some very deep theorems, it is possible to show in the case that (X, d) is Euclidean that, when it is finite, the length defined here matches the length in Equation 1.1. Part of the use of deep theorems is the fact that we can still use Equation 1.1 even if γ has a lot (even infinitely many) points of non-differentiability!

3. Finsler length of a curve in $\mathbb{R}^2$

We will now show a new way to define lengths of curves in $\mathbb{R}^2$. In full generality, Finsler geometry works on any *smooth manifold*, but we will not be discussing manifolds! We will give a specialized version of the definition of Finsler metric which works for $\mathbb{R}^n$, but needs a little work to generalize further. For simplicity, when it is clear from context we will generally suppress the overhead arrow when discussing vectors or points in $\mathbb{R}^n$.

Definition 3.1. A *Finsler metric* on $\mathbb{R}^n$ is a continuous function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with the following properties for all $(x, v), (x, v') \in \mathbb{R}^n \times \mathbb{R}^n$:

(1) $F(x, v) \ge 0$ with equality if and only if $v = 0$
(2) $F(x, \lambda v) = \lambda F(x, v)$ for all $\lambda \ge 0$.
(3) $F(x, v + v') \le F(x, v) + F(x, v')$

This might seem very strange at first! But here is how to think about it. In the domain $\mathbb{R}^n \times \mathbb{R}^n$, you should think of the first copy as the usual set of points of $\mathbb{R}^n$. You should think of the second copy as the set of vectors based at the point in the first coordinate. For example, in $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$, x is a point in the space, and v is the vector based at x pointing in some direction. If you traveled from x along v, you would end up at the point $x + v$ (though this is mixing vectors with points, so it is a little weird to write it this way).

Exercise 3.2. See if you can come up with some example(s) of Finsler metrics on $\mathbb{R}^n$. Even trivial ones (whatever that means) are a good start! Maybe even try with $n = 1$. Draw some pictures of vectors based at points and think about how the Finsler metric behaves for different ways of manipulating the vectors.

Exercise 3.3. Prove that the Euclidean length of vectors is a Finsler metric on $\mathbb{R}^n$ (maybe just do the case of $n = 1$ and $n = 2$). Note that the associated function F is constant in the first input.

Why do we want such a function? Why is it called a Finsler "metric"? Think of F as a way to measure the *lengths* of vectors. So, the length of the vector v based at the point x is $F(x, v)$. This isn't a metric in the sense of Worksheet 1, because it does not directly give distances between points. However, recall from Equation 1.1 that if we had some notion of length of vectors tangent to a curve, we could compute its length.

Here is another challenging exercise, but it is hard for different reasons!

Exercise 3.4. Given a smooth curve $\gamma : [0, 1] \to \mathbb{R}^n$ and a Finsler metric $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, give a reasonable definition of the *length with respect to the Finsler metric* F of γ. This should be some formula which one could compute using a parametrization of γ.

Hint: This might resemble Equation 1.1. Compare to Exercise 3.3.

Finsler metrics are used as a way to define lengths of curves in spaces with many different kinds of metrics. Not all distance-function metrics come from Finsler metrics, but all Finsler metrics induce a (possibly asymmetric) metric. It will be easier to compute a Finsler length than the length using the rectifiable curves approach in the previous section. Fortunately, as we will see, PFMs are all Finsler metrics (... hence the name)!

One can also use a Finsler metric to define a distance function (metric, in the sense of Worksheet 1): just define the infimum of lengths of curves between two points to be the distance!

Exercise 3.5. Write the definition of a distance function in symbols using the above description.

Exercise 3.6. Combine Exercises 3.5, 3.3, and 1.3 to prove that the distance function defined by the Finsler metric associated to the Euclidean length of vectors is a metric on $\mathbb{R}^n$. Prove that it is equal to the Euclidean metric.

Note: I haven't worked this out myself so I don't know how hard it is! Just spend enough time to make sure you understand the question and what it would mean to provide such a proof.

4. Polyhedral Finsler metrics on $\mathbb{R}^2$

Now, we want to be able to measure lengths of curves in $\mathbb{R}^2$ endowed with PFMs rather than just the Euclidean metric. Write $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. Given a convex polygon P containing the origin in $\mathbb{R}^2$, we want to see how to define F so it is a Finsler metric for the PFM induced by P. It may not even be clear that PFMs should be Finsler! Our best approach might be to try to define a Finsler metric that works, and show that it does.

Exercise 4.1. Think about what properties of PFMs will make this task easier. Maybe we only need to define it at one point?

Exercise 4.2. Show that if $F(x, v)$ is constant in x and $\gamma : [0, 1] \to \mathbb{R}^2$ is a line segment with constant derivative, then

$$\ell_F(\gamma) = F(0, \gamma'(0)) \cdot \ell_{\mathbb{R}^2}(\gamma).$$

Think about what is going on here and why this should is the case.

Exercise 4.3. What is the best definition of F for a PFM defined by polygon P?

Hint: This is definitely a challenge. Please ask for more hints/discussion if you find yourself very confused! Try to draw a lot of pictures and compute some possible examples, even if they seem too easy. Maybe even review PFMs for $\mathbb{R}^1$. Definitely don't spend much time on these last few if you did not complete the exercises in sections 1 and 3.

APPENDICES

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There are a few loose ends in this project, as with any. Here, we have a few proofs written up, as well as a list of future work at the end. Perhaps a future Log(M) project could continue this work!

1. Checking the definition of PFM

Recall first the definition of polyhedral Finsler metric, and the associated Finsler norm on $T(\mathbb{R}^n)$, the tangent bundle of $\mathbb{R}^n$.

Definition 1.1. A *Polyhedral Finsler metric* on $\mathbb{R}^n$ is a Finsler metric whose distance function is determined by a prescribed convex polytope $P \subseteq \mathbb{R}^n$ consisting of all points at a unit (forward) distance from 0. The (forward) distance $d(p, q)$ between two points $p, q \in \mathbb{R}^n$ is defined by

$$d(p, q) = \lambda \text{ such that } q - p \in \partial(\lambda P)$$

where λP is the radius λ ball about 0, given by dilating P by a factor of λ. In particular, the polytope P is the unit ball.

The definition above only prescribes an (asymmetric) metric (rather than a function on the tangent bundle). However, such a function may be obtained as follows. First, note that defining it at the origin in $\mathbb{R}^n$ will suffice, since parallel translation of the norm is just translation along Euclidean lines. Define the length of a (Euclidean) unit vector based at 0 to be the reciprocal of the Euclidean distance to the unit ball P in that direction. One can verify that this defines a Finsler metric with the appropriate distance function for the given unit ball. An alternative (but equivalent) way is utilized in earlier worksheets.

We will assume only that P is a convex closed subset of $\mathbb{R}^n$ with the origin as an interior point.

Proposition 1.2. *Let $P \subseteq \mathbb{R}^n$ be a compact convex set with 0 as an interior point. Then using the definition of d in Definition 1.1, we obtain a (possibly asymmetric) metric on $\mathbb{R}^n$.*

Proof. Positive-definiteness is immediate since 0 is an interior point. For the triangle inequality, let $a, b, c \in \mathbb{R}^n$. We must show that

$$d(c, b) \leq d(c, a) + d(a, b).$$

Observe first that it will suffice to assume c is the origin, since translations are isometries. Further, we need only show it in the plane, since any three points in higher dimensions exist in a unique plane, and a planar slice through the origin is simply a restriction of the function d. Further, we may assume $d(0, a) = 1$, since scaling preserves all relevant properties. We write $d(a, b) = \epsilon$; then $b \in (\epsilon P + a)$. We must show that $b \in (1 + \epsilon)P$. See the picture below.

Notice that $(1+\epsilon)a \in (1+\epsilon)P$ since $a \in P$ and $\frac{1+\epsilon}{\epsilon}(b-a) \in (1+\epsilon)P$ since $b-a \in \epsilon P$. A convex combination of these two vectors yields b:

$$\frac{1}{1+\epsilon}(1+\epsilon)a + \frac{\epsilon}{1+\epsilon}\frac{1+\epsilon}{\epsilon}(b-a) = b$$

and so by convexity of P, we have $b \in (1+\epsilon)P$, as required. $\square$

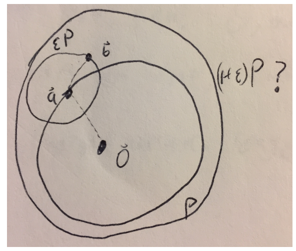

FIGURE 1. We show that the largest ball, $(1+\epsilon)P$, contains the point b, which is in $\epsilon P + a$ (Finsler distance ϵ from a), where $a \in P$ is in the unit ball about 0.

2. Boxes of geodesics

Next, we give the "recipe" for generating the union of geodesics (which we might call a "box" of geodesics) between arbitrary pairs of points, where "geodesic" just refers to distance-minimizing paths (continuous images of the unit interval) between specified endpoints. The recipe and its proof are made precise here. Recall that a metric space is said to be *geodesic* if between any two points there exists a (not necessarily unique) geodesic. Denote by $B_L(p,r)$ the closed "left" ball of all points $d(p,x) \leq r$ and $B_R(p,r)$ the closed "right" ball of all points $d(x,p) \leq r$. For asymmetric metrics these will not be equal.

Proposition 2.1. *Let X be a geodesic metric space, possibly asymmetric. Let $a, b \in X$, and write $D = d(a,b)$. Then the following sets are equal:*

$$\bigcup \{\gamma([0,1]) : \gamma \textit{ is a geodesic from } a \textit{ to } b\} = \bigcup_{0 \leq r \leq D} B_L(a,r) \cap B_R(b, D-r)$$

Proof. Let $p \in B_L(a,r) \cap B_R(b, D-r)$ for some $r \in [0,D]$. Let γ_1 be a geodesic from a to p, and γ_2 a geodesic from p to b. They are necessarily length r and $D-r$, respectively. Then the concatenation $\gamma_1 \cup \gamma_2$ is a curve of length D from a to b (and hence necessarily a geodesic) containing p, as required.

Let $q \in \gamma([0,1])$ for some geodesic γ from a to b. Note that γ must be distance-minimizing between all pairs of points in its image. So, if $d(a,q) = s$, then $d(q,b) = D-s$, since q is on a distance-minimizing path in the forward direction from a to b. But this means $q \in B_L(a,s) \cap B_R(b, D-s)$ for some $0 \leq s \leq D$, as required. □

This approach has a nice visual interpretation by shrinking and growing the left s- and right $D-s$-ball around a and b respectively and looking at where the boundaries meet up. This viewpoint also enables one to show uniqueness of geodesics for metrics with some kind of strict convexity condition.

The usual unit ball P coming from the definition of the PFM is the left ball $B_L(0,1)$. Computing the right ball $B_R(0,1)$ would allow one to use Proposition 2.1 in explicit examples. It seems that $B_R(0,1) = -B_L(0,1)$, but a proof of this remains to be written.

3. Some conjectures and suggestions for future work

Here are a few directions to go for further study. Several of these are statements which we are confident are true but remain unproven, or whose proofs need some cleaning up. Some of these are ideas for additional programs to write or more vague project directions to go in. Some of these are expected to be easy, others not so much!

(1) Prove several facts about the length formula

$$\ell(\gamma) = \int_0^1 d_P(0, \gamma'(t))dt$$

where $\gamma : [0,1] \to \mathbb{R}^2$ is a continuously differentiable curve and d_P is a PFM associated to a convex polygon $P \subseteq \mathbb{R}^2$ with the origin in the interior. In particular:
- Show $\ell(\gamma) \geq d_P(\gamma(0), \gamma(1))$ (this together with a quick calculation can show straight lines are always distance-minimizing).
- Show $\ell(\gamma) = \lim_{n\to\infty} \sum_{k=0}^{n} d_P\big(\gamma(k/n), \gamma((k+1)/n)\big)$, that is, the length of γ found by using equally-sized pieces matches this Finsler metric formula. This might be hard in sufficient generality. Maybe replace the right-hand side with the intrinsic length of γ as a rectifiable curve (seen in a previous worksheet).

(2) Show $B_R(0,1) = -B_L(0,1)$ (see "Boxes of geodesics" above).
(3) Show d_P is symmetric if and only if $B_R = B_L$. This is not expected to be hard.
(4) Show that the length-minimizing curve from the origin to a vertex of P is unique (only a straight Euclidean line). This should follow easily from Proposition 2.1.
(5) Is the above condition both necessary and sufficient?
(6) Characterize paths which are length-minimizing in one or both directions. This might be doable using Proposition 2.1. We expect there to be some condition on the tangent vectors $\gamma'(t)$ such as requiring them to be within a range of angles.
(7) Carefully generalize some of the definitions and calculations (or just check that they work) for piecewise-differentiable curves.
(8) Write a program to draw boxes of geodesics using Proposition 2.1.
(9) Do everything in this project for dimension 3. Or for dimension n.
(10) PFMs show up in some other research areas (Hilbert metrics on convex domains, apparently?). Maybe some of what is done here can be applied there.