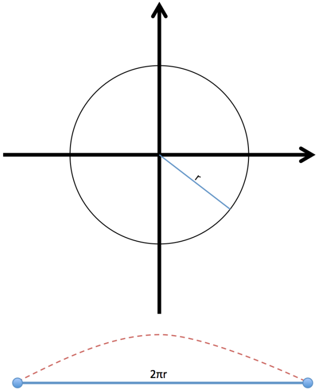
Geometry of n-tori: donuts in all dimensions

Mark Greenfield
University of Michigan, Ann Arbor. Email: markjg@umich.edu

May 6, 2020
(Virtual) MC2 Colloquium, Kalamazoo College

What is a circle?

Set of points equidistant from given centerpoint. Equation: $x^2 + y^2 = r^2$.

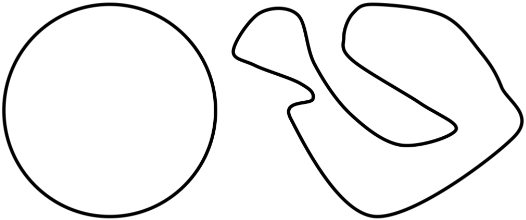


Equivalently: given line segment length $2\pi r$: glue ends together. The endpoints become a *single point*!

Tori in dimension 1: circles

Essential features of a circle, for an ant living on it:

- Can only go forwards or backwards (dimension)
- Walk in a straight line, end up at starting point (topology)
- Characterize by distance to reset at starting point (geometry)



To an ant walking on either object, these are the same 1-dimensional space!

Let's take these features up a dimension....

Digression #1: What is “dimension”?

Calculus: (x, y)-plane is 2D, (x, y, z)-space is 3D
Linear algebra: Size of a basis of a vector space

Unifying idea: How many numbers (coordinates) needed to specify a point on the space

Examples:

- *Surface* of planet Earth: Latitude and longitude: 2 coordinates, 2 dimensions
- Any simple closed loop: Angle coordinate $[0, 2\pi)$: 1 coordinate, 1 dimension
- Point in spacetime? (x, y, z, t) for spatial and time dimensions: 4 coordinates, 4 dimensions (general relativity)

Tori in dimension 2

At each point, ant sees a tiny plane-like area (like surface of Earth). Two special directions lead it back to the starting point in a short time (forward/backward, left/right). Two coordinates should be able to specify a point.

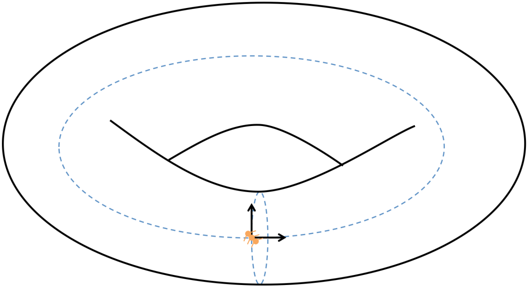


Figure: Coordinates: $[0, 2\pi) \times [0, 2\pi)$ for position on *each* of the two independent loops.

How do we characterize the size? What about the other directions? How can we represent this as a mathematical object?

Dimension 2: “hollow” tori

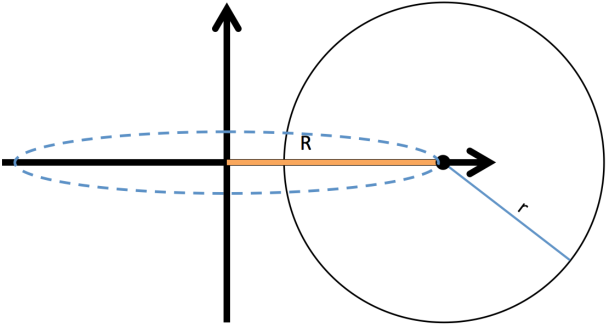
Keep in mind: the torus is “hollow”: it is just the glaze, not the whole donut! Including the dough would make it a *solid torus*.

Figure: Three solid tori. The dough makes it a solid torus, the glaze itself is a 2-torus. Image credit: krispykreme.com.

Constructing a 2-dimensional torus: in $\mathbb{R}^3$

Living in $\mathbb{R}^3$: surface of revolution for a circle away from the axis.

Equation: $\left(\sqrt{x^2+y^2}-R\right)^2+z^2=r^2$.



We will not use this perspective again!

Constructing a 2-dimensional torus: intrinsically

Let's capture the intrinsic behavior (*2* special directions return you to starting point) using the "gluing" method.

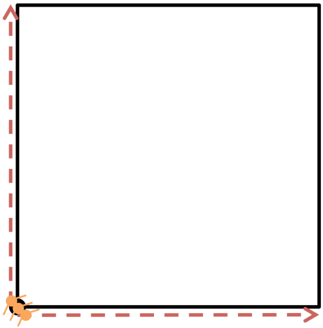
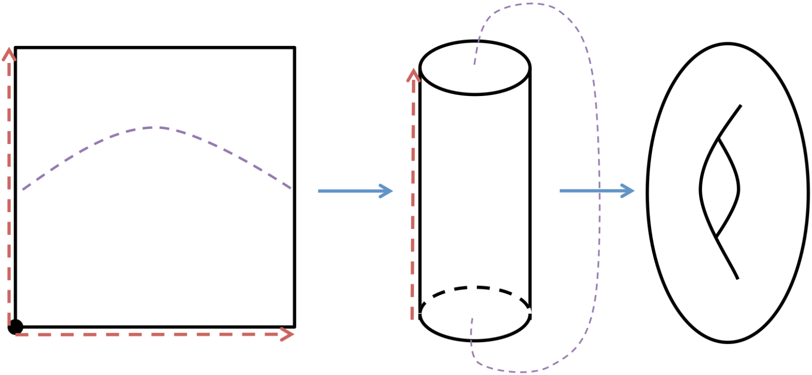


Figure: Start with a square for 2-torus instead of the line segment we used for the 1-torus.

If ant goes up or right, it returns to the starting point. So... glue opposite edges together!

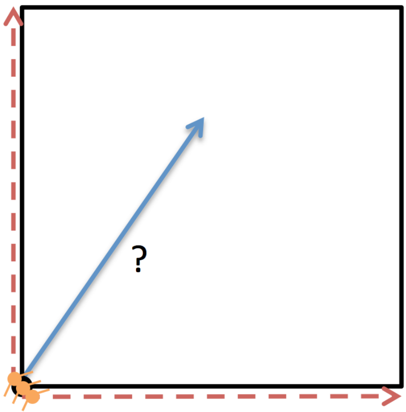
Folding the 2-torus from a square



This is a standard construction in geometric topology. Similar tools can construct *all possible* 2-dimensional surfaces.

Traveling in other directions on the torus

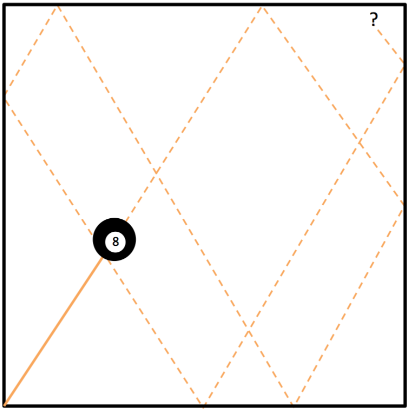
What if the ant travels in some other direction?



Will it return to the starting point?

Digression #2: Billiards and dynamics

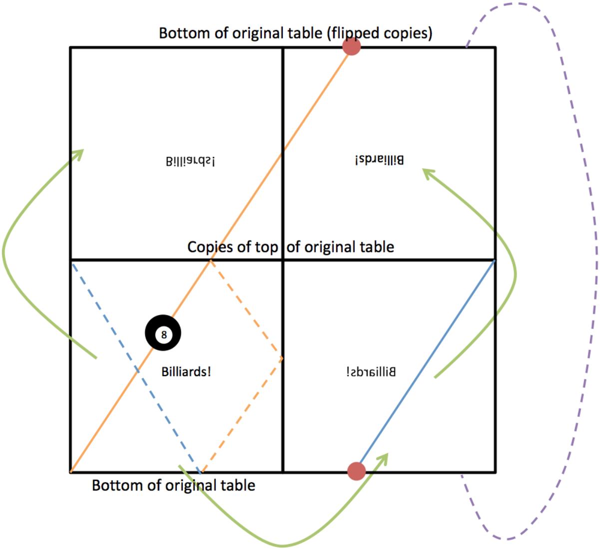
Classical problem: billiards on a unit-side square table (for physicists: with perfectly elastic collisions and no friction). What kinds of paths are possible?



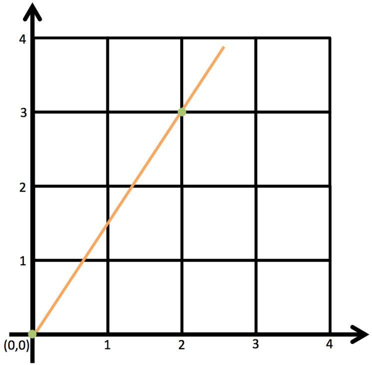
Quick observation: some paths repeat, such as perfect horizontal or vertical.

Unfolding the billiards table

Turn billiard paths into torus paths with this ONE NEAT TRICK!



Back to the torus paths - how do we know what it will do?



Torus paths are characterized by their *slope*. Each square is an unfolded copy of the whole torus.

Similar trick: draw the unfolded square (torus) as a repeated lattice!

Recall the gluing: all "lattice points" (integer coords) are glued together.

Starting at $(0, 0)$: path is periodic iff associated line hits integer point.

Which paths are periodic?

Periodic torus path $\leftrightarrow$ lines $y = mx$ hitting integer point

Proposition

The line $y = mx$ in $\mathbb{R}^2$ hits an integer point other than $(0, 0)$ if and only if the slope is rational.

Proof.

1. "$\Leftarrow$": If $m = p/q$, then (q, p) is an integer point on line.
2. "$\Rightarrow$": If (q, p) is on the line, then the slope is $m = (p - 0)/(q - 0) \in \mathbb{Q}$.

$\square$

What about the irrational slopes?

If not periodic, where do they go? They fill it up!

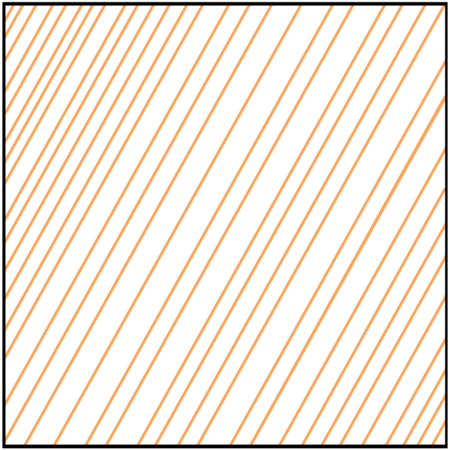
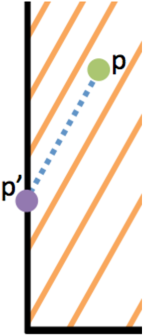


Figure: The orange lines never overlap (same slope, different intercepts), there are infinitely many of them, and the path goes on forever....

Dense torus paths

Proposition

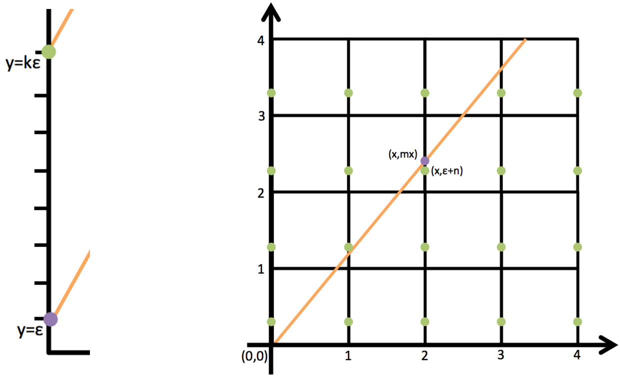
Let m be the slope of an irrational torus path γ. Then for every point p in the torus, γ comes arbitrarily close to p.



Slide p down a line of slope m to p' the x-axis. If γ comes close to p', then it comes close to p.

We may assume the point has an integer x-coordinate (the above observation shows this special case proves the whole result).

Completing the proof: an outline



If at an integer x the path reaches ϵ above some integer, then the path goes within ϵ of some copy of p. We need this for arbitrarily small ϵ. Equivalently: there are $n, r \in \mathbb{Z}$ with $mr - n < \epsilon$.

This is Dirichlet's approximation theorem (classical analytic number theory; uses Pigeonhole Principle).

The classical dichotomy

We have demonstrated the following famous result in dynamics:

Theorem

For every straight-line trajectory on the square torus, exactly one of the following holds:

- *The slope is rational and the path is periodic*
- *The slope is irrational and the path is dense*

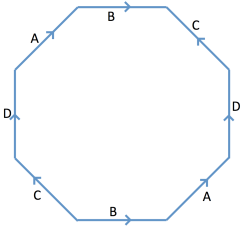
Exercise: Extend this result to rectangular tori... or even parallelograms!

Corollary

The same dichotomy holds for billiard paths.

Digression #3: Translation surfaces

Dynamics on polygons with sides glued together (maybe in a complicated way) is the topic of *translation surfaces*, which connects to Teichmüller theory and hyperbolic geometry. Much of Maryam Mirzakhani's Fields medal work was also in this area. In her language, we would say *orbit closures of the straight-line flow on the 2-torus are trivial.*

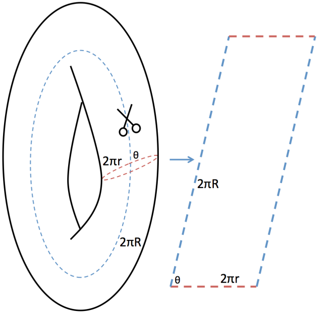


Exercise: (1) Try to figure out what kind of surface this represents. (2) Following the gluings, what do some possible paths look like? Are they all either periodic or dense?

The space of possible 2-tori

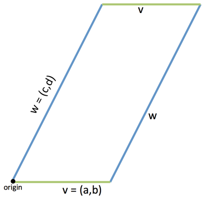
Question: What are all 2-dimensional tori? How do we find/characterize them?

- Find two "geodesic" loops (not necessarily perpendicular)
- Cut along curves, must get quadrilateral (one pair of sides for each curve)
- Must be parallelogram (Prove it!)
- Classify tori by parallelograms!



The space of parallelograms

- Put the bottom-left corner at the origin
- Parallelogram determined by two *linearly independent* vectors $v = (a, b), w = (c, d)$
- Write as a 2×2 matrix! Invertible since lin. indep.
- What about rotations? Same torus, so rotation matrix should not change it



$$\mathrm{GL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mid \det \neq 0 \right\}$$

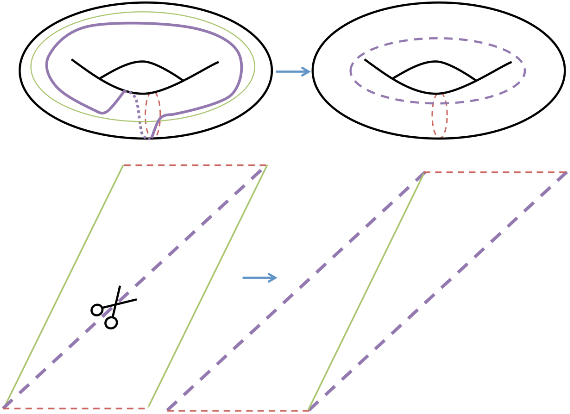
$$\mathrm{O}(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$$

$$\{\text{all parallelograms}\} = \mathrm{GL}(2, \mathbb{R})/\mathrm{O}(2)$$

But wait... we had to choose the pair of curves!

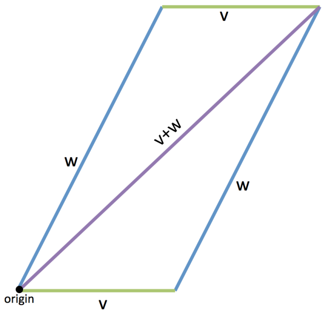
The curve-choice problem

There is no *intrinsic* difference between these two realizations of the *same* torus. Same paths, same lengths, same angles.



This is called a *scissors congruence*. Let's think about this using linear algebra....

Scissors congruence in terms of matrices



- If $v = (a, b), w = (c, d) \in \mathbb{R}^2$ are vectors for the sides, then diagonal is $v + w$.
- So $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \sim \begin{pmatrix} a+c & c \\ b+d & d \end{pmatrix}$ represent the same torus
- Claim 1: Any basis of $\mathbb{R}^2$ built from v, w has the same problem (scissors congruent). (Why? Sequence of diagonal cuts!)
- Claim 2: All such bases are obtained by 2×2 integer matrix similarity: $A \sim XAX^{-1}$ for $A \in \mathrm{GL}(2, \mathbb{R})$, $X \in \mathrm{GL}(2, \mathbb{Z})$

The moduli space of 2-tori

We may now write down the space of all 2-tori.

Definition

The *moduli space of 2-dimensional tori* is given by $\mathrm{GL}(2,\mathbb{Z})\backslash\mathrm{GL}(2,\mathbb{R})/\mathrm{O}(2)$.

Usually, different area tori may be considered the same (just scale it up or down, no meaningful difference). Claim: this is realized by restricting all matrices to determinant 1.

So, *the moduli space of unit area 2-tori* is

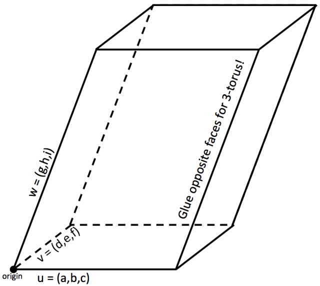
$$\mathcal{M}_2 = \mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2).$$

Moduli spaces (spaces classifying all objects of a certain type) are fundamental objects of study across mathematics, especially algebraic geometry. Example: The moduli space of lines through the origin in $\mathbb{R}^{k+1}$ is called *projective space* $\mathbb{RP}^k$.

Higher dimensions

Everything so far extends to higher dimensions. Our work so far makes it relatively easy! Replacements:

- Parallelepipeds instead of parallelograms
- Cut and unroll with low-dimensional tori instead of loops
- Scissors congruence is cutting by a plane instead of a line
- 2×2 matrices becomes $n \times n$
- $\mathcal{M}_n = \mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$



Some missing pieces: Curvature

- These are all *flat* tori. A flat n-torus needs $2n$-dimensional ambient space! (by the *Nash embedding theorem* - same Nash as *Nash equilibria* and the book/movie *A Beautiful Mind*)
- A real donut has points of positive, negative, and zero *curvature*, but it averages to zero. Cosmologists wonder what the average curvature of the universe is! General relativity describes the curvature of spacetime.
- Gauss-Bonnet theorem says any constant-curvature torus must be flat. What about *higher genus* surfaces? Negative curvature and hyperbolic geometry!

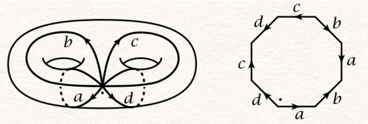


Figure: Image credit: Hatcher, A. *Algebraic Topology*.

Some missing pieces: Different perspectives

- Sometimes we care about the choice of basis (cutting curves). Then we have *Teichmüller space* $\mathcal{T}_n = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$
- The space $\mathcal{M}_n = \mathrm{SL}(n, \mathbb{Z})\backslash\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ is a *rank* $n-1$ *locally symmetric space* associated to the *Lie group* $\mathrm{SL}(n, \mathbb{R})$.
- Topology: the circle (1-torus) is denoted S^1. The n-torus is the *product space* $(S^1)^n$ (higher tori = product of lower tori)
- The n-torus is $\mathbb{T}^n = \mathbb{R}^n/\Lambda$, the quotient of the lattice $\Lambda \cong \mathbb{Z}^n$ acting on $\mathbb{R}^n$ by vector addition
- $\mathrm{SL}(n, \mathbb{Z})$ is the *mapping class group* of the n-torus. Mapping class groups are central in manifold topology and Teichmüller theory. For other surfaces, mapping class groups are dramatically more complicated.

Thank you!

Introductory reading on topology, geometry, manifolds, and Lie theory, accessible after multivariable calculus and linear algebra:

- Gamelin, T., and Greene, R. *Introduction to topology.*
- Do Carmo, M. *Differential geometry of curves and surfaces.*
- Spivak, M. *Calculus on manifolds.*
- Hall, B. *Lie groups, Lie algebras, and representations.*

Two relevant recent papers:

- My research on spaces of n-tori: Greenfield, M., Ji, L. *Metrics and compactifications of Teichmüller spaces of flat tori.* Available at `https://arxiv.org/abs/1903.10655`.
- Survey on billiards and related topics: Wright, A. *From rational billiards to dynamics on moduli spaces.* Available at `https://arxiv.org/abs/1504.08290`.

Advanced references (available free online) on topology, geometry, and moduli spaces of surfaces:

- Hatcher, A. *Algebraic Topology*
- Farb, B., and Margalit, D. *A primer on mapping class groups.*