1 January 28: Eric Canton, Introduction to (F)rigid geometry

References include:

(1) (characteristic 0) Jonsson–Mustață, “Valuations and asymptotic invariants of sequences of ideals”.

(2) (characteristic 0) Boucksom–de Fernex–Favre–Urbinati, “Valuation spaces and multiplier ideals on singular varieties”.

(3) (characteristic $p$) Canton, “Berkovich log discrepancies in positive characteristic”.

For today, $X$ will be an excellent integral (separated) scheme over a field $k$, and point will not mean closed points. We write $L = \text{Frac} X$.

**Definition.** (1) A (real) valuation $v$ on $L$ is centered on $X$ if there’s some $x \in X$ such that $O_{X,x} \subset A_v := \{ f \in L : v(f) \geq 0 \}$.

(2) If $v$ is centered on $X$, there’s a unique $x \in X$ such that $O_{X,x} \subset A_v$ is local (i.e., $m_v \cap O_{X,x} = m_x$); this $x \in X$ is called the center of $v$ on $X$, written $c_X(v)$.

(3) $\text{Val}(X) = \{ \text{valuations } v \text{ on } L \text{ centered on } X \}$.

**Remark.** If $X$ is a projective (or proper) over $k$, then every valuation on $L$ is centered on $X$. If $X$ is not projective, then this is not true: for example, if $X = \text{Spec} k[t]$ is the affine line over $k$, then the valuation $\text{ord}_t$ is not centered on $X$.

**Remark.** If $R \subset A_v$, then $m_v \cap R \in \text{Spec} R$; call this point $p$. Then $R_p \subset A_v$ is a local inclusion and $p = c_{\text{Spec} R}(v)$.

**Example** (examples of $v \in \text{Val}(X)$). (1) Say $X$ is normal (or just regular in codimension 1); then if $E \subset X$ is a prime divisor then $O_{X,E}$ is a DVR and the associated valuation $\text{ord}_E : L^\times \to \mathbb{Z}$ is in $\text{Val}(X)$. For $c \in (0, \infty)$, $c \cdot \text{ord}_E : L^\times \to c \cdot \mathbb{Z}$ is another point of $\text{Val}(X)$.

(2) If $\pi : Y \to X$ is a birational morphism and $Y$ is normal, then for any $E \subset Y$ a prime divisor on $Y$ and every $c > 0$ we have $c \cdot \text{ord}_E \in \text{Val}(X)$ (note that this is centered on $X$, and $\eta$ is the generic point of $\pi(E) \subset X$, then $c_X(c \cdot \text{ord}_E) = \eta$ for all $c > 0$).

(3) The trivial valuation $\text{triv}_X : L^\times \to \{0\}$ is a point of $\text{Val} X$, centered at the generic point of $X$ (this will be important as a sort of “limit point” of the valuation space).
(4) Suppose \( R = K[[x_1, \ldots, x_d]] \); pick \( r \in (\mathbb{R}_{>0})^d \). Then for \( f = \sum c_\alpha x^\alpha \), with \( c_\alpha \in K \), we define

\[
\text{val}_L(f) := \min \left\{ \sum_{i=1}^d r_i \alpha_i : c_\alpha \neq 0 \right\}.
\]

Jonsson–Mustaţă, Proposition 3.1, showed that val is well-defined, and that this process gives an injective map \( \text{val} : (\mathbb{R}_{>0})^d \to \text{Val}_{\text{Spec } R}, r \mapsto \text{val}_L \).

**Remark.** In characteristic 0 at least, we can realize the valuation space as a limit of these cones over resolutions of our variety.

(5) Suppose \( \pi : Y \to X \) is birational and \( Y \) is regular, and \( H = \bigcup_{i=1}^{t} H_i \) is an SNC divisor on \( Y \) (with each \( H_i \) a regular irreducible hypersurface). For \( m \leq n \), consider the irreducible decomposition \( \bigcap_{i=1}^{n} H_i = Z_1 \cup \cdots \cup Z_l \). Let \( \epsilon \) be the generic point of \( Z_1 \). Then \( \mathcal{O}_{Y, \epsilon} \) is a regular local ring of dimension \( d \), and \( \hat{\mathcal{O}}_{Y, \epsilon} \) is isomorphic to \( \mathcal{O}(\eta)([z_1, \ldots, z_d]) \); we can choose \( z_i \) to be a local equation for \( H_i \).

Now, as in the preceding example, we can define \( \text{val}_H : (\mathbb{R}_{>0})^d \to \text{Val}_{\hat{\mathcal{O}}_{Y, \epsilon}} \); moreover, we can restrict the valuation to \( \mathcal{O}_{Y, \epsilon} \subset \hat{\mathcal{O}}_{Y, \epsilon} \). We can restrict further along

\[
\mathcal{O}_{X, \pi(\eta)} \to \mathcal{O}_{Y, \epsilon} \to \hat{\mathcal{O}}_{Y, \epsilon}.
\]

So, we get an actual point of \( \text{Val} X \) in this manner. We can thus consider \( \text{im}(\text{val } H) \subset \text{Val} X \), and we denote this by \( \text{QM}_n(Y, H) \). We call these quasimonomial valuations.

**Remark.** Quasimonomial valuations are always Abhyankar valuations; the converse is true in characteristic 0, or at least as long as \( k \) is perfect. We’ll discuss this more next time.

(6) Let \( R = k[x_1, \ldots, x_d] \), and let \( A = k[[t]] \). Then \( \text{Frac } A = k((t)) \) has infinite transcendence degree over \( k \) (Exercise: is it true that the transcendence degree is the cardinality of the power set of the natural numbers?). Choose \( d-1 \) transcendental series \( f_2, \ldots, f_d \in k[[t]] \) that are algebraically independent and without constant terms. Define \( \varphi : R \to k[[t]] \) where \( \varphi(x_1) = t, \varphi(x_j) = f_j \) for \( j \geq 2 \). This is injective by construction; define \( V = \text{ord}_L \varphi \). This is a \( \mathbb{Z} \)-valuation of \( \text{Spec } R \), centered at \( (x_1, \ldots, x_d) \). We’ll show next time that this \( v \) doesn’t come from \( \text{ord}_E \) for any \( \pi : Y \to \mathbb{A}_d^1 \), or even as a quasimonomial valuation. (In fact, this will fail to be an Abhyankar valuation.)

Now, we put a topology on \( \text{Val} X \). There are three equivalent ways to define this topology, which we’ll do in decreasing order of abstraction:

1. We have a set-theoretic inclusion \( \text{Val} X \to \bigcap_{f \in L^\times} (-\infty, \infty), v \mapsto \bigcap_{f \in L^\times} v(f) \), where the right side carries the product topology. We can then give \( \text{Val} X \) the subspace topology with respect to this inclusion.

2. Each \( f \in L^\times \) gives a function \( \hat{f} : \text{Val} X \to \mathbb{R} \), where \( \hat{f}(v) = v(f) \) (\( \hat{f} \) is “like the Gelfand transform of \( f \)”). Now, give \( \text{Val} X \) the coarsest topology such that \( \hat{f} \) is continuous for all \( f \in L^\times \).

3. Fix \( v \in \text{Val} X \). A basis for the topology near \( v \) is given as follows: for any \( s \), choose \( f_1, \ldots, f_s \in L^\times \) and \( \epsilon > 0 \), and define

\[
\mathcal{U}(f_1, \ldots, f_s; \epsilon) := \bigcap_{i=1}^s \left\{ w \in \text{Val} X : \left| \frac{\hat{f}(v) - \hat{f}(w)}{w(f_i) - v(f_i)} \right| < \epsilon \right\}.
\]

**Remark (facts).** (1) \( \text{Val} X \) is Hausdorff in this topology.

(2) \( c_X : \text{Val} X \to X \) is anticontinuous (the preimage of a closed set is open and vice versa).
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Recall from last time that we claimed the center map \( c_X \) is anticontinuous. A “proof”: Let \( X = \text{Spec} R \) and \( U = D(f) \). Then \( v \in \text{Val} X \) if and only if \( R \subseteq A_v \), so \( v(f) \geq 0 \). Moreover, \( c_X(v) \in U \) if and only if \( f \notin m_v \) if and only if \( v(f) = 0 \). Since \( \hat{f} : \text{Val}_X \to \mathbb{R} \) is continuous, so \( \hat{f}^{-1}(0) \) is closed, but this is \( c_X^{-1}(U) \).

(“Proof” is in quotes because we reduced to the affine case; we need to check that \( \text{Val}_X \) is actually covered by finitely many closed sets of this form.)

Quasimonomial valuations and retractions

Recall that given a birational map \( \pi : Y \to X \) from a regular scheme \( Y \) (not necessarily proper!) and \( H = \sum_i H_i \) an snc divisor on \( Y \). We write \( [N] = \{1, \ldots, N\} \), for \( J \subset [N] \) we write \( H_J := \bigcap_{j \in J} H_j \) (if \( J = \emptyset \) then \( H_J = Y \)). If \( H_J \) is nonempty than we can decompose \( H_J \) into disjoint irreducible components \( Z_1 \cup \cdots \cup Z_l \), and at each generic point \( \eta_i \) of \( Z_i \) we’ve defined \( \text{QM}_{\eta_i}(Y, H) \cong (\mathbb{R}_{\geq 0})^d \). The \( \eta_i \) that appear as \( J \) ranges over all subsets of \( [N] \) are called the strata of \( H \). For \( y \in Y \), we write \( J_y := \{ j \in [N] : y \in H_j \} \); then for any \( y \) we have \( y \in H_{J_y} \), for a unique \( J \). We let \( \eta(y) \) be the associated strata. Thus we have a function from points of \( Y \) to strata.

Note that if \( \eta \) is a stratum then \( \eta = \eta(\eta) \), so we don’t risk ambiguity in this notation.

Definition. (1) \( \text{QM}_y(Y, H) := \text{QM}_{\eta(y)}(Y, H) \).

(2) \( \text{QM}(Y, H) := \bigcup_y \text{QM}_y(Y, H) \subset \text{Val} X \).

We now define the retraction map. Assume that our map \( \pi : Y \to X \) is actually proper. Then every \( v \in \text{Val}_Y \) has center on \( Y \) (this is the valuative criterion for properness).

Remark. Note that in the proof the valuative criterion for properness, we consider not just valuations on the function field of a variety, but on proper closed subschemes as well. Thus properness is much stronger than valuations on \( X \) having center on \( Y \); what it really means is that if we compactify \( \text{Val}_X \) by including the valuations on proper closed subschemes (yielding the Berkovich space \( X^\text{B} \)) that these have centers on \( Y \) as well.

Now, let \( v \in \text{Val}_X \), \( y = c_Y(v) \). Let \( D \) be a (Cartier) divisor on \( Y \), such that near \( y \) we have \( D = \text{div} f \) for \( f \in \text{Frac} X \). We define \( v(D) = v(f) \). Given an snc divisor \( H = \sum H_i \) on \( Y \), we write \( a_i = v(H_i) \); note that \( a_i > 0 \) if and only if \( y \in H_i \). We thus have \( \text{QM}_y(Y, H) = \text{QM}_{\eta(y)}(Y, H) \), with \( \eta(y) \) the generic point of some component of \( \bigcap_H \{ J : a_j > 0 \} H_j \).

Definition. We define \( r_{\pi(Y, H)}(v) = \text{val}(a_1, \ldots, a_l) \), where we renumber such that \( a_1, \ldots, a_l > 0 \) and the rest are zero. (Alternatively, we can say \( r_{\pi(Y, H)}(v) = \sum v(H_i) \text{ord}_H \).)

We’ve thus defined a map \( r_{\pi(Y, H)} : \text{Val}_X \to \text{QM}(Y, H) \) for every snc pair \( \pi : (Y, H) \to X \) (with \( \pi \) proper and birational), such that:

(1) If \( v \in \text{QM}(Y, H) \) then \( r_{\pi(Y, H)}(v) = v \).

(2) \( r_{\pi(Y, H)} \) is continuous.

These are called the monomialization retractions.

Inverse systems

Suppose \( \mu : Y' \to Y \) is another proper birational map sitting over \( X \), and we have snc divisors \( H' \) and \( H \) on \( Y' \) and \( Y \) respectively. Suppose furthermore that \( \text{Supp}(\mu^* H) \subset \text{Supp}(H') \).

Lemma. The diagram

\[
\begin{array}{ccc}
\text{Val}_X & \xrightarrow{r_{\pi(Y, H)}} & \text{QM}(Y, H) \\
r_{\pi(Y', H')} \downarrow & & \downarrow r_{\pi(Y, H)_{|\text{QM}(Y', H')}} \\
\text{QM}(Y', H') & & \text{QM}(Y', H')
\end{array}
\]
commutes.
Moreover, each cone $(\mathbb{R}_{\geq 0})^r \subset \text{QM}(Y', H')$ maps to a single cone $(\mathbb{R}_{\geq 0})^r \subset \text{QM}(Y, H)$ via a matrix of nonnegative integers $b_{ij}$.

**Proof.** Because of our assumption on the supports of $H'$ and $H$, we can write $\mu^*(H_i) = \sum_{j=1}^{N} b_{ij} H'_j$, with $H = \sum_{i=1}^{M} H_i$ and $H' = \sum_{j=1}^{N} H'_j$. Moreover, $b_{ij} \geq 0$. Now, if $v \in \text{Val}_X$, then $v(H_i) = \sum b_{ij} v(H'_j)$, so that we have $r_{(Y,H)}(r_{(Y',H')}(v)) = r_{(Y,H)}(v)$.

The point of all of this is the following:

**Theorem (JM 4.9).** Assume $\mathbb{Q} \subset k$. Then the natural map $r : \text{Val}_X \rightarrow \text{lim}_{(Y,H)} \text{QM}(Y, H)$ is a homeomorphism.

The main idea in proving this is the following: we want to show that $v(D) = r_{(Y,H)}(v)(D)$ whenever $(Y, H) \rightarrow (X, D)$ gives a log resolution; thus, evaluating any divisor $D$ under $v$ can be done via the retract to a quasimonomial valuation on a log resolution. This is the only place we need characteristic 0: we need the existence of log resolutions of $(X, D)$.

**Log discrepancies**

Say $X$ is a normal variety over $k$ and $\Delta$ is a $\mathbb{Q}$-Weil divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : Y \rightarrow X$ be a proper birational map, with $Y$ regular and $H$ an snc divisor on $Y$. For all divisors $E$ on $Y$ we have the log discrepancy $A_X(E, \Delta) = 1 + \text{ord}_E(K_Y - \pi^*(K_X + \Delta))$. For all $v \in \text{QM}(Y, H)$, we can define $A_X(v; \Delta)$ to be $\sum v(H_i) A_X(H_i; \Delta)$.

See Proposition 5.1 of Jonsson and Mustata for the verification that this is well-defined (independent of our choice of model $(Y, H)$). The takeaway is that $A_X(-; \Delta)$ is well-defined and affine-linear (in particular, continuous) on each cone in $\text{QM}(Y, H)$. Thus, we can define (in characteristic 0) $A_X(v, \Delta) = \sup_{(Y,H)} A_X(r_{(Y,H)}(v), \Delta)$, and $A_X(-; \Delta) : \text{Val}_X \rightarrow \mathbb{R} \cup \{\infty\}$ is lower-semicontinuous. The lower-semicontinuity is still true in characteristic $p$, although the construction and proof are different. From these log discrepancies one can define multiplier ideals, and use the topology on the valuation space to show that these are actually coherent ideals (which is new in characteristic $p$, see Canton).