Motivic integration

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1 Introduction

The theory of motivic integration began in 1995 with Kontsevich's announcement of a proof of the birational invariance of Hodge numbers for Calabi–Yau manifolds.

A theorem of Batyrev

Recall that the i-th Betti number of a smooth complex variety X is $b_i(X) := \dim H^i(X, \mathbb{C})$. We say a smooth projective complex variety is *Calabi–Yau* if $\omega_X := \bigwedge^{\dim X} \Omega_X$ is trivial, i.e., if $\omega_X \cong \mathcal{O}_X$. (In differential geometry other conditions may be imposed, but this general definition suffices for our purposes.)

Theorem ([Bat99])**.** *Let X_1, X_2 be birationally equivalent Calabi–Yau varieties. Then $b_i(X_1) = b_i(X_2)$ for all i.*

Example. Let's just note briefly we need the Calabi–Yau condition, i.e., that we don't expect the dimensions of cohomology groups to be a birational invariant. One can calculate easily that if $Y := \mathrm{Bl}_p X \to X = \mathbb{P}^2$, then

$$H^2(Y, \mathbb{C}) = H^2(X, \mathbb{C}) \oplus \mathbb{C},$$

so $b_2(Y) = b_2(X) + 1$.

Exercise. Check this calculation, using the topological description of blowing up the plane and Mayer–Vietoris.

Thus the triviality of ω_X is really the important ingredient in Batyrev's theorem!

Sketch of proof. To treat Batyrev's proof fully, we'd need to develop the theory of p-adic integration; we'll attempt to sketch the main ideas here as motivation for what follows.

Given two birationally equivalent varieties X_1, X_2, we can dominate both by a common model Y, with Y smooth projective and the morphisms $Y \to X_i$ proper and birational:

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 \end{tikzcd}$$

Thinking of Y, X_1, X_2 and all morphisms involved as being defined over some finitely generated extension of $\mathbb{Z}$, we get $\mathbb{Z}$-schemes

$$\begin{tikzcd} & Y_{\mathbb{Z}} \arrow[dl] \arrow[dr] & \\ (X_1)_{\mathbb{Z}} & & (X_2)_{\mathbb{Z}} \end{tikzcd}$$

By the Weil conjectures, the Betti numbers of X_1 can be calculated by knowing the cardinality of $X_1(\mathbb{F}_{p^n})$ for all n, and likewise for X_2. But then one can turn this point counting into the calculation of a certain p-adic integral on $X_1(\mathbb{Q}_p)$ (respectively $X_2(\mathbb{Q}_p)$); one then uses the change-of-variables formula for p-adic integration to calculate this integral via p-adic integration on $Y(\mathbb{Q}_p)$, with a factor in the integrand corresponding to order of vanishing along the determinant of the Jacobian of the map $Y \to X_1$ (respectively $Y \to X_2$). Finally, the fact that X_1, X_2 are Calabi–Yau implies that this factor is the same in both integrals on Y, and thus the point counts and Betti numbers of X_1 and X_2 agree. $\square$

Kontsevich's Orsay lecture

In 1995, Kontsevich announced a generalization of the above theorem of Batyrev. Recall that the Hodge numbers of a smooth projective variety are defined as

$$h^{p,q}(X) = \dim H^q(X, \Omega_X^p).$$

Equivalently, the Hodge decomposition says that the cohomology groups $H^i(X, \mathbb{C})$ decompose as

$$\bigoplus_{p+q=i} H^{p,q}(X)$$

for certain canonical vector subspaces $H^{p,q}(X) \subset H^i(X, \mathbb{C})$; then $h^{p,q}(X) = \dim H^{p,q}(X)$.

Theorem (Kontsevich 1995)**.** *Let X_1, X_2 be birationally equivalent Calabi–Yau varieties. Then $h^{p,q}(X_1) = h^{p,q}(X_2)$ for all p, q.*

The proof of this theorem, and the motivation for motivic integration, is a generalization of the use of p-adic integration above: their are two essential ideas:

(1) Rather than controlling cohomological information by keeping track of the point counts of a variety, we should instead keep track of "the class of the variety", in a suitable sense.

(2) Moreover, point-counting corresponded to an integral on the $\mathbb{Q}_p$-valued points of a variety, which necessitated first passing from our complex variety to a $\mathbb{Z}$-model; instead, we replace the mixed-characteristic DVR $\mathbb{Z}_p$ by $\mathbb{C}[[t]]$, and integrate on $X(\mathbb{C}[[t]])$, the so-called "arcs" on X.

This minicourse will develop the ideas necessary to prove Kontsevich's result as follows:

(1) We'll begin by developing the theory of the Grothendieck ring of varieties, which is the "value ring" that motivic integration takes values in.

(2) We'll then discuss arc schemes, which will be the underlying "measure space" for motivic integration.

(3) We'll construct an algebra of "measurable" sets in the arc space, and thus define motivic integration.

(4) We'll state and potentially prove the birational transformation rule, which will allow us to relate motivic integrals on birational varieties.

(5) Finally, we'll prove Kontsevich's theorem, and time permitting give other applications of the theory of motivic integration.

2 The Grothendieck ring of varieties

In this section, we largely follow the exposition in [CLNS18].

Let S be an arbitrary scheme (with no finiteness conditions imposed).

Definition (Grothendieck ring of varieties)**.** Let $K_0(\mathrm{Var}_S)$ be the abelian group generated by symbols $e(X)$ for finite-type S-schemes X, modulo the "scissor" relations

$$[Z] + [X - Z] = [X]$$

whenever $Z \subset X$ is a finitely presented closed subscheme. Note that $e(\emptyset)$ is the zero element.

The multiplicative structure is defined by extending

$$e(X) \cdot e(Y) := e(X \times_S Y)$$

via $\mathbb{Z}$-linearity; note that $e(S)$ is the unit element.

Exercise. Check that the multiplication is well-defined.

In the case where S is noetherian, which is the only one we'll treat, the finite presentation is automatic, so we'll effectively ignore it from now on. We write $\mathbb{L}$ for the class of the affine line $e(\mathbb{A}^1_S)$.

Example. For intuition, let $S = \operatorname{Spec} k$, although this is unnecessary.

(1) Since $\mathbb{A}^n_k = \mathbb{A}^1_k \times_k \cdots \times \mathbb{A}^1_k$, we have that

$$e(\mathbb{A}^n_k) = e(\mathbb{A}^1_k) \cdots e(\mathbb{A}^1_k) = \mathbb{L}^n.$$

(2) Since $\operatorname{Spec} k$ is a closed subset of $\mathbb{P}^1_k$, with complement the open set $\mathbb{A}^1_k$, we have that

$$e(\mathbb{P}^1) = e(\mathbb{A}^1) + e(\operatorname{Spec} k) = \mathbb{L} + 1.$$

(3) Similarly, we know that $\mathbb{P}^n$ contains a closed subscheme isomorphic to $\mathbb{P}^{n-1}$, with complement $\mathbb{A}^n$, we have that

$$e(\mathbb{P}^n) = e(\mathbb{A}^n) + e(\mathbb{P}^{n-1}) = \mathbb{L}^n + e(\mathbb{P}^{n-1});$$

by induction, we conclude that this is

$$\mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + \mathbb{L} + 1.$$

Remark. If $k = \mathbb{F}_q$ is a finite field, note that the expressions in (1) through (3) resemble the number of k-points of $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$, except with $\mathbb{L}$ instead of rq. We can thus think of the point counts on $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ as being obtained by specializing the classes $e(\mathbb{A}^n_k)$ and $e(\mathbb{P}^n_k)$ by setting $\mathbb{L}$ to q. Much more on this shortly!

(4) Let X be a smooth variety and $Z \subset X$ a smooth closed subvariety. Let

$$f : Y = \mathrm{Bl}_Z X \to X$$

be the blowup of X along Z, with exceptional divisor E. Note that f is an isomorphism $Y - E \to X - Z$, so that $e(Y - E) = e(X - Z)$. Moreover, we have that $e(Y) = e(Y - E) + e(E)$ and $e(X) = e(X - Z) + e(Z)$. Putting these together, we have

$$e(Y) - e(E) = e(X) - e(Z).$$

This may seem tautological, but we'll see shortly that in characteristic 0 all relations on $K_0(\mathrm{Var}_k)$ arise this way!

(5) Say $f : X \to Y$ is an "F-fibration", by which we mean a morphism $X \to Y$ such that each fiber $(f^{-1}(y))_{\mathrm{red}}$ is isomorphic to $F \times_k \kappa(y)$. The main example to keep in mind is a vector bundle or projective bundle. Then one can show via induction that $e(X) = e(Y)e(F)$ in $K_0(\mathrm{Var}_k)$. In particular, in the preceding example, we know that $E = \mathbb{P}(\mathbb{N}_{Z/X})$, so that $e(E) = e(Z)e(\mathbb{P}^{c-1}) = e(Z)(\mathbb{L}^{c-1} + \cdots + \mathbb{L} + 1)$, where $c = \operatorname{codim}(Z, X)$.

Remark. Let S be noetherian. Recall that a constructible subset E of a scheme X is a finite union of locally closed subsets. One can show that there is a well-defined class $e(E)$ in the Grothendieck ring of varieties, behaving exactly as one would expect (e.g., $e(X) = e(E) + e(X - E)$). This will be necessary for our purposes later, but is straightforward so we omit the verifications.

Remark. Note that the finite-type condition is quite natural after some thought: if we don't impose this condition, let X be the infinite disjoint union of copies of S. Then $X \sqcup S$ is isomorphic to X, of course, so $e(X) + e(S) = e(X)$, and thus $e(S) = 0$; since $e(S) = 1$, however, this collapses the entire ring to 0.

Generators and relations for $K_0(\mathrm{Var}_S)$

We specialize now to the case where S is noetherian; S-schemes will be assumed to be of finite type. We first want to see that one can restrict to "reasonable" schemes as generators of $K_0(\mathrm{Var}_S)$.

Remark. Note that if S is noetherian and X is a finite-type S-scheme, then the reduced subscheme $X_{\mathrm{red}} \subset X$ is defined by a finitely presented ideal sheaf. Since X and X_{red} have the same underlying topological space, $X - X_{\mathrm{red}}$ is empty, and the scissor relations say that $e(X) = e(X_{\mathrm{red}})$. Thus, any class in $K_0(\mathrm{Var}_S)$ can be expressed as the combination of classes of *reduced* S-schemes.

Note that as a consequence, if $U \subset X$ is open, then the class of the complement of U is independent of the closed subscheme structure we choose.

Remark. Let X be a reducible S-scheme. We want to show that $e(X)$ can be expressed as the sum of classes of irreducible S-schemes. Since X has finitely many irreducible components, by induction we may assume X has two irreducible components, say X_1, X_2. Then $e(X) = e(X_1) + e(X_2) - e(X_1 \cap X_2)$ by the scissor relations; since $\dim X_1 \cap X_2 < \max(\dim X_1, \dim X_2)$, by induction we may express $e(X_1 \cap X_2)$ as the sum of classes of irreducible S-schemes, and thus we're done.

Thus, we see that $K_0(\mathrm{Var}_S)$ is generated by the classes of integral S-schemes. Moreover, if X is an integral S-scheme, then there's some open affine $U \subset X$. By the same argument as above, we can write $e(X) = e(U) + e(\text{lower dimension})$, and thus express the class of any S-scheme via classes of affine (or quasiprojective) varieties. Note also that this shows one can use only classes of separated varieties.

Remark. If S is excellent (i.e., most of the time we'll care about), one can similarly show that $K_0(\mathrm{Var}_S)$ is generated by the classes of *regular* integral S-schemes; similarly, if $S = \operatorname{Spec} k$ for k a perfect field, $K_0(\mathrm{Var}_k)$ is generated by the classes of smooth integral S-schemes. These are proved exactly as above: we have regularity/smoothness on some open subset (by excellence/perfectness) and thus by induction on dimension we obtain the result.

Bittner's theorem

When S is a field of characteristic 0, we can be explicit not only about the generators of S, but the relations among the generators.

Theorem ([Bit04])**.** *Let $k = \mathbb{C}$. Then $K_0(\mathrm{Var}_{\mathbb{C}})$ is generated by the classes of smooth irreducible projective varieties. Moreover, if K is the free abelian group on isomorphism classes of smooth irreducible projective varieties, the kernel of the surjection $K \to K_0(\mathrm{Var}_{\mathbb{C}})$ is generated by relations*

$$[\mathrm{Bl}_Y X] - [E] = [X] - [Y],$$

where $Y \subset X$ are smooth projective varieties, with E is the exceptional divisor of $\mathrm{Bl}_Y X$.

We note that we can assign K a ring structure, again via products, and this map is actually a ring homomorphism.

Proof. We show only the generation. First, we show by induction that for an irreducible variety X of dimension n, $[X]$ is equal to $[Y] + \sum m_i [W_i]$ for Y, W_i smooth projective irreducible, Y birational to X, and $\dim[W_i] < n$.

The key ingredient is Hironaka's resolution of singularities, which says there is a birational map $Y \dashrightarrow X$ with Y smooth projective irreducible. Let $U \subset X$ and $V \subset Y$ be open sets where this is an isomorphism, so $[U] = [V]$. Then $[X - U]$ and $[Y - V]$ are each of dimension $< n$, hence by induction are equal to the sum of classes smooth projective irreducible varieties. Thus we have that

$$[X] = [X - U] + [U] \quad \text{and} \quad [Y] = [Y - V] + [V],$$

so

$$[X] - [Y] = [X - U] - [Y - V] = \sum m_i [F_i].$$

Since $[Y]$ is itself smooth projective irreducible, by induction the result follows.

We're not quite done: we only have the result for irreducible varieties, but if $X = X_1 \cup X_2$, X_1, X_2 closed, we don't have $[X] = [X_1] + [X_2]$. So, let X be arbitrary; we induct again on $\dim n$. Say $X = \bigsqcup X_i$ with X_i irreducible. Let $Y_i = X_i - \left(\bigcup_{j \neq i} X_j\right)$, and $Y = \bigcup Y_i$ (the elements that are in a unique X_i). This is open in X, so $X - Y$ is of lower dimension than n, so by the inductive hypothesis it's fine. Y itself is also fine, since $[Y] = \sum [Y_i]$ and each Y_i is is irreducible. $\square$

We omit the description of the kernel, but the key ingredient is the following weak factorization theorem:

Theorem ([AKMW02])**.** *Say* $\operatorname{char} k = 0, \bar{k} = k$. *If*

$$\varphi : X \dashrightarrow X'$$

is a birational map of smooth projective varieties, then φ is the composition of blowups and blowdowns of smooth projective varieties along smooth irreducible centers, i.e., we can write φ as the composition

$$\begin{tikzcd}
 & X_1 \arrow[dl] \arrow[dr] & & X_3 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & \\
X = X_0 & & X_2 & & X_4 & & X_m = X'
\end{tikzcd}$$

where each X_i is smooth projective and each $X_i \to X_{i-1}$, $X_i \to X_{i+1}$ is the blowup at a smooth center.

Moreover, there's some index i such that

(1) for all $j \leq i$ the birational maps $X_j \dashrightarrow X$,

(2) for all $j \geq i$ the birational maps $X_j \dashrightarrow X'$,

are in fact morphisms.

Remark. One way to remember the second result of the theorem, or why one would care, is to state what a "strong factorization" is: If $X \dashrightarrow X'$ is a birational map of smooth projective varieties, a strong factorization is a factorization of this birational morphism

$$\begin{tikzcd}
 & & & X_i \arrow[dl] \arrow[dr] & & & \\
 & & \cdots \arrow[dl] & & \cdots \arrow[dr] & & \\
 & X_1 \arrow[dl] & & & & X_{m-1} \arrow[dr] & \\
X = X_0 & & & & & & X_m = X'
\end{tikzcd}$$

where every map is a blowup of a smooth variety along smooth center. In a strong factorization, the second condition is immediate.FK

Motivic invariants

The utility of $K_0(\mathrm{Var}_S)$ is that it is a "universal" invariant among those satisfying the scissor relations. We can formalize this as follows: let A be a ring, and let

$$\rho : \{S\text{-schemes}\} \to A$$

be a function such that

(1) $\rho(S) = 1$.

(2) $\rho(X) = \rho(X - Z) + \rho(Z)$ for any S-scheme X and closed subscheme Z of X.

(3) $\rho(X \times_S Y) = \rho(X)\rho(Y)$ for any X, Y.

We call such a function an A-valued motivic invariant on S-schemes. It's clear essentially by definition that given an A-valued motivic invariant, there is a unique map $\widetilde{\rho} : K_0(\mathrm{Var}_S) \to A$ such that

$$\widetilde{\rho}(e(X)) = \rho(X)$$

for all S-schemes X; moreover, one can check that this is a universal property characterizing $K_0(\mathrm{Var}_S)$.

Thus, $K_0(\mathrm{Var}_S)$ should be thought of as a "universal motivic invariant".

Example. Let $k = \mathbb{F}_q$ be a finite field. Then point counting, e.g., $|X(\mathbb{F}_q)|$ is a motivic invariant on k-schemes. As we've seen already, the corresponding map $K_0(\mathrm{Var}_k) \to \mathbb{Z}$ is obtained by sending $e(X)$ to $|X(\mathbb{F}_q)|$.

Cohomology theories and motivic invariants

Let $S = \operatorname{Spec} k$. Depending on what k is, there are a variety of cohomology theories one can apply to k-varieties (e.g., étale cohomology, or if $k = \mathbb{C}$ one can use singular cohomology on the associated complex analytic space, or if k is a perfect field of positive characteristic one can use crystalline cohomology). In fact, for technical reasons one should use cohomology with compact support $H_c^*(-)$. We claim that for a reasonable cohomology theory the assignment

$$\chi : X \mapsto \sum (-1)^i [H_c^i(X)]$$

gives a motivic invariant on k-schemes, where the image lives in the Grothendieck group of k-vector spaces often with some additional structure.

"Suitable" cohomology theories $H_c^*(-)$ have a Mayer–Vietoris sequence

$$\cdots \to H_c^{m-1}(A \cap B) \to H_c^m(X) \to H_c^m(A) \oplus H_c^m(B) \to H_c^m(A \cap B) \to \cdots$$

for subspaces A, B covering X. In particular, if one takes $A = Z$ and $B = X - Z$ for Z a closed subset, we have that

$$H_c^m(Z) \oplus H_c^m(X - Z) \cong H_c^m(X).$$

This then implies that χ respects the scissor relations. Furthermore, the Künneth formula implies that χ is multiplicative (i.e., $\chi(X)\chi(X') = \chi(X \times_k X')$), and thus χ is indeed a motivic invariant. We'll discuss explicitly one particularly important case, that of the singular cohomology of $\mathbb{C}$-varieties.

The Hodge–Deligne polynomial

Recall that if X is a smooth projective (or just proper) complex variety that the Hodge numbers $h^{p,q}(X) := \dim H^q(X, \Omega_X^p)$ are finite, and we have that

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X).$$

One can encode this information via a polynomial in two variables, the so-called Hodge–Deligne polynomial:

$$\mathrm{HD}(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) u^p v^q \in \mathbb{Z}[u, v].$$

By [Del71a, Del71b, Del74], we can in fact assign *any* $\mathbb{C}$-variety X a polynomial in $\mathbb{Z}[u, v]$, still denoted $\mathrm{HD}(X)$, in a way that agrees with the definition above for X smooth and projective. Moreover, this assignment satisfies

(1) $\mathrm{HD}(X) = \mathrm{HD}(Z) + \mathrm{HD}(X - Z)$.

(2) $\mathrm{HD}(X)\,\mathrm{HD}(Y) = \mathrm{HD}(X \times Y)$.

We thus get an induced map

$$\overline{\mathrm{HD}} : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[u, v]$$

such that $\overline{\mathrm{HD}}(e(X)) = \mathrm{HD}(X)$ for any $\mathbb{C}$-variety X. This will be key to our proof of Kontsevich's theorem.

Remark. One more key fact we'll need is that $\deg \mathrm{HD}(X) \le 2 \dim X$.

Remark (the details)**.** We've included a brief sketch of how the above map $\overline{\mathrm{HD}}$ is obtained as a specialization of a more general motivic invariant; the theory involved is somewhat technical, so one may take the existence of the motivic invariant $\overline{\mathrm{HD}}$ on faith (or use [AK06] and [Bit04] to extend the definition from the case of a blowup of a smooth projective variety along a smooth center).

Definition (mixed Hodge structure)**.** A (rational) mixed Hodge structure $(M, F^\bullet, W_\bullet)$ consists of the following data:

(1) a $\mathbb{Q}$-vector space M;

(2) a decreasing exhaustive finite filtration $F^\bullet$ on $M_{\mathbb{C}} := M \otimes_{\mathbb{Q}} \mathbb{C}$, called the Hodge filtration;

(3) an increasing exhaustive finite filtration $W_\bullet$ on M, called the weight filtration;

such that for every n we have $\mathbb{C}$-subspaces $M^{p,q} \subset \mathrm{gr}_n^W(M)_{\mathbb{C}}$, for $p + q = n$, such that:

(1) $\overline{M^{p,q}} = M^{p,q}$.

(2) For each n, $\bigoplus_{p+q=n} M^{p,q} = \mathrm{gr}_n^W(M)_{\mathbb{C}}$.

(3) For each n, the induced filtration from $F^\bullet$ on $\mathrm{gr}_n^W(M)_{\mathbb{C}}$ is given by

$$F^p \,\mathrm{gr}_n^W(M)_{\mathbb{C}} = \bigoplus_{p' \ge p} M^{p', n-p'}.$$

"Intuitively", this just says that the filtration induced by $F^\bullet$ on the n-th associated graded of $W_\bullet$ is a pure Hodge structure of weight n, i.e., behaves like the Hodge decomposition of the n-th cohomology of a smooth projective variety.

Remark. One can check that the conditions above uniquely specify the subspaces $M^{p,q} \subset \mathrm{gr}_n^W(M)_{\mathbb{C}}$. This is familiar from the case of the Hodge structure on the cohomology of a smooth projective variety, where the data of the Hodge filtration and action by complex conjugation is equivalent to the Hodge decomposition.

Definition (Hodge–Deligne polynomial)**.** Given a mixed Hodge structure $(M, F^\bullet, W_\bullet)$, we write $h^{p,q}(M) = \dim M^{p,q}$. We define the Hodge–Deligne polynomial

$$\mathrm{HD}(M) = \sum h^{p,q}(M) u^p v^q \in \mathbb{Z}[u, v, u^{-1}, v^{-1}].$$

One defines morphisms of mixed Hodge structures as $\mathbb{Q}$-linear maps compatible with both filtrations. By [Del71b], this makes mHS, the category of mixed Hodge structures into an abelian category. It's easy to check then that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of mixed Hodge structures, we have

$$\mathrm{HD}(M) = \mathrm{HD}(M') + \mathrm{HD}(M'').$$

Given mixed Hodge modules M, N, there is a natural mixed Hodge structure on $M \otimes N$, where we take:

- $W_n(M \otimes N)$ is the subspace generated by $W_m(M) \otimes W_{m-n}(N)$ for all $m \in \mathbb{Z}$.
- $F^n(M \otimes N)$ is the intersection of the subspaces $\ker\big((M \otimes N)_{\mathbb{C}} \to (M_{\mathbb{C}}/F^m M_{\mathbb{C}}) \otimes (N_{\mathbb{C}}/F^{n-m} N_{\mathbb{C}})\big)$ for all $m \in \mathbb{Z}$.

One can then check that $\mathrm{HD}(M \otimes N) = \mathrm{HD}(M)\,\mathrm{HD}(N)$.

Definition. Let $K_0(\mathrm{mHS})$ be the Grothendieck ring of the abelian category of mixed Hodge structures; this is the free abelian group on symbols $[M]$ for M a mixed Hodge structure, modulo the relation

$$[M] = [M'] + [M'']$$

when we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

As a result, we have that the Hodge–Deligne polynomial extends to a ring morphism

$$K_0(\mathrm{mHS}) \to \mathbb{Z}[u, v, u^{-1}, v^{-1}].$$

Definition (polarized Hodge structure)**.**

Theorem (Deligne)**.**

One can then define $\mathrm{HD}(X)$ by composing the map $K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{pMHS})$ with $\mathrm{HD} : K_0(\mathrm{pMHS}) \to \mathbb{Z}[u, v]$.

Other realizations

We have similar results for other cohomology theories:

(1) Let k be a field and fix a separable closure k^s of k; let $G_k = \mathrm{Gal}(k^s/k)$. Fix ℓ a prime invertible in k. Étale cohomology gives a motivic invariant valued in the Grothendieck group of continuous finite-dimensional $\mathbb{Q}_\ell$-vector spaces with a continuous G_k-action. Then, for example, the étale Euler characteristic is obtained via further specialization from this Grothendieck group (by applying the dimension function).

(2) Let k be a finite field, let $W(k)$ be the Witt vectors of k (or the field of fractions of any DVR with residue field k) and let $K = \mathrm{Frac}\, W(k)$. Write F for the lift of the Frobenius on k to V and K. An F-isocrystal on $K$is a finite-dimensional K-vector space M with an injective F-linear morphism $F_M : M \to M$. The theory of crystalline cohomology (with compact supports) assigns to a k-variety X rigid cohomology groups $H^i_{\mathrm{crys,c}}(X)$ with an F-isocrystal structure, which then gives a motivic invariant taking value in the Grothendieck group of F-isocrystals.

(3) Let $k = \mathbb{C}$. One can show that the fundamental group is a motivic invariant in the following sense: let fpGrp be the set of isomorphism classes of finitely presented groups, which is a monoid under group product; let $\mathbb{Z}^{\mathrm{fpGrp}}$ be the associated monoid ring. One can show (see [CLNS18, Example 6.2.3]) that there's a unique ring morphism $K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}^{\mathrm{fpGrp}}$ sending $e(X)$ to $\pi_1(X)$ for every X smooth projective over $|C$. Thus, we have that if two smooth projective complex varieties have the same class in $K_0(\mathrm{Var}_k)$, they have the same fundamental groups. This construction depends on the below theorem of [LL03] on the relation between stable birationality classes and the Grothendieck group.

Localization and completion

We begin by localizing $K_0(\mathrm{Var}_S)$; the need for this will arise in our attempt to give a meaningful motivic class to certain subsets of *non-finite-type* schemes.

Definition. Given any scheme S, write $\mathcal{M}_S = K_0(\mathrm{Var}_S)[\mathbb{L}_S^{-1}]$.

Given an S-scheme of finite type X, we'll write $e(X)/1$ to denote the image of $e(X)$ in $\mathcal{M}_S$. As we'll see shortly, if $S = \operatorname{Spec} k$ for k a field of characteristic 0, $\mathbb{L}$ is a zerodivisor in $K_0(\mathrm{Var}_k)$, so the canonical map

$$K_0(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)[\mathbb{L}_k^{-1}] = \mathcal{M}_k$$

is not injective (and we have no reason to expect this to change for more general S). Thus it will be important to distinguish between $e(X)$ and $e(X)/1$.

Remark. By combining the universal properties of ring localization and of $K_0(\mathrm{Var}_S)$, it's immediate that $\mathcal{M}_S$ is specified by the following universal property: giving a ring map $\mathcal{M}_S \to A$ is the same as giving a motivic invariant

$$\rho : \{S\text{-varieties}\} \to A$$

such that $\rho(\mathbb{A}^1_S)$ is a unit.

Example. Consider the Hodge–Deligne polynomial; we know that this is a motivic invariant $\mathrm{HD} : \{\mathbb{C}\text{-varieties}\} \to \mathbb{Z}[u, v]$. One can check that $\mathrm{HD}(\mathbb{A}^1) = uv$, so that the Hodge–Deligne polynomial, viewed as a motivic invariant taking values in $\mathbb{Z}[u, v, (uv)^{-1}]$, extends to $\mathcal{M}_{\mathbb{C}}$.

In order to give certain asymptotic quantities meaning, we'll need to complete $\mathcal{M}_S$ as follows:

Definition. Given an S-variety X, we write $\dim(X/S) = \max \dim X_s = X \times_S \operatorname{Spec} \kappa(s)$ for s a point of X, where $\kappa(s)$ is the residue field of S.

The main case we'll consider is $S = \operatorname{Spec} k$ for k an algebraically closed field, in which case $\dim(X/\operatorname{Spec} k)$ is just the usual dimension.

Definition. We define a decreasing exhaustive $\mathbb{Z}$-filtration on $\mathcal{M}_S$ as follows: for each $d \in \mathbb{Z}$, we let $F^d\mathcal{M}_S$ be the subgroup of $\mathcal{M}_S$ generated by $(e(X)/1) \cdot \mathbb{L}^{-p}$ for $\dim(X/S) - p \leq -d$.

Exercise. Check that this makes $\mathcal{M}_S$ into a filtered ring (i.e., that $(F^p\mathcal{M}_S)\cdot(F^q\mathcal{M}_S) \subset F^{p+q}\mathcal{M}_S$).

Definition. We define $\widehat{\mathcal{M}}_S = \varprojlim_d \mathcal{M}_S/F^d\mathcal{M}_S$, the completion of $\mathcal{M}_S$ with respect to the "dimension" filtration above. There is thus a canonical morphism $\mathcal{M}_S \to \widehat{\mathcal{M}}_S$; we write $\overline{\mathcal{M}}_S$ for the image of $\mathcal{M}_S$ in $\widehat{\mathcal{M}}_S$.

Remark. We'll need to understand when we extend a motivic invariant ρ from $\mathcal{M}_S$ to $\overline{\mathcal{M}}_S$. By definition the map $\mathcal{M}_S \to \overline{\mathcal{M}}_S$ is surjective; thus a motivic invariant can be extended from $\mathcal{M}_S$ exactly when it takes value 0 on the kernel of this map, which is $\bigcap F^d\mathcal{M}_S$. Unfortunately, it's unknown if this kernel is nonzero, even for $S = \operatorname{Spec} k$. Thus all we can do is rephrase this condition as demanding that if $e(X)/1 \in F^d\mathcal{M}_S$ for all d then $\rho(X) = 0$.

Example. Let $S = \operatorname{Spec} \mathbb{C}$ and consider the Hodge–Deligne polynomial. Let $\mathfrak{a}$ be a class in $F^d\mathcal{M}_S$. Then $\mathfrak{a}$ is the sum of classes of the form $e(X)/\mathbb{L}^p$, for $\dim X - p \leq -d$, so that $\mathrm{HD}(\mathfrak{a})$ is the sume of $\mathrm{HD}(e(X)/\mathbb{L}^p)$. For each of these classes, we have that $\mathrm{HD}(e(X)/\mathbb{L}^p) = \mathrm{HD}(e(X))/(uv)^p$. By our above remark, we know that $\deg \mathrm{HD}(e(X)) \leq 2\dim X$. Thus the degree of $\mathrm{HD}(e(X))/(uv)^p$ is $2\dim X - 2p$, which by assumption is $\leq -2d$. But then $\mathrm{HD}(\mathfrak{a})$ has degree $\leq -d$ for all d, so $\mathrm{HD}(\mathfrak{a})$ must be 0, and thus the Hodge–Deligne polynomial factors through $\overline{\mathcal{M}}_S$.

Remark. In fact, the motivic measure $\chi : K_0(\mathrm{Var}_k) \to K_0(\mathrm{pMHS})$ is itself separated (i.e., factors through $\overline{\mathcal{M}}_S$), by a similar argument; thus, any invariant obtained by specializing from $K_0(\mathrm{pMHS})$ will be as well.

Similarly, one can check that the étale motivic measure (taking values in the Grothendieck ring of finite-dimensional G_k-representations on $\mathbb{Q}_\ell$-vector spaces) is separated as well, so all the invariants obtained through étale cohomology will also factor through $\overline{\mathcal{M}}_S$.

Remark. As we'll see later, motivic integration will take values in $\widehat{\mathcal{M}}_S$; this is because we'll need to assign motivic "classes" in an asymptotic way. As such, the formalism of motivic integration is best suited to prove things about motivic invariants factoring through $\overline{\mathcal{M}}_S$.

Exercise ([Bli11])**.** Show that for any $s \in \mathbb{N}$

$$\sum_i^{\infty} \mathbb{L}^{-is} = \frac{1}{1 - \mathbb{L}^{-s}}$$

in $\overline{\mathcal{M}}_S$ (and that we can thus manipulate elements of $\overline{\mathcal{M}}_S$ like formal power series).

Pathologies of the Grothendieck group

Even for $S = \operatorname{Spec} k$, the rings $K_0(\mathrm{Var}_k)$, $\mathcal{M}_k$, and $\widehat{\mathcal{M}}_k$ are quite subtle object, and much remains unknown or challenging about their structure:

(1) It is not known if the completion map $\mathcal{M}_k \to \widehat{\mathcal{M}}_k$ is injective; thus, we may lose information upon passing to $\widehat{\mathcal{M}}_k$.

(2) $K_0(\mathrm{Var}_k)$ fails to be a domain; in fact, $\mathbb{L}$ is a zerodivisor! By [Bor18] it's known over a field k of characteristic 0 that there are smooth projective Calabi–Yau threefolds, with $e(X) \neq e(Y)$, such that $(e(X) - e(Y))\mathbb{L}^6 = 0$. The fact that $K_0(\mathrm{Var}_k)$ fails to be a domain goes back in fact to [Poo02], who constructs abelian varieties A, B such that $e(A) \neq \pm e(B)$, but with $A \times A \cong B \times B$ and thus $e(A)^2 = e(B)^2$, so that $(e(A) - e(B))(e(A) + e(B)) = 0$.

(3) In fact, $K_0(\mathrm{Var}_k)$ is not even reduced, as shown by [Eke10] using a variant of Poonen's result.

(4) In characteristic 0, [LS10] showed that if $\{A_i\}$ is a collection of complex abelian varieties with $\mathrm{Hom}(A_i, A_j) = 0$, then the elements $\{e(A_i)\}$ are algebraically independent. By noting that the cardinality of simple isogeny classes of abelian varieties is the same as the cardinality of k, we have that $K_0(\mathrm{Var}_k)$ has infinitely many algebraically independent elements (or even uncountably many when $k = \mathbb{C}$!) and thus is not noetherian.

Remark. All these complexities reflect in some sense the rich information contained in the Grothendieck ring of varieties. We give just one example here, first recalling a definition:

Definition. Let X and Y be k-varieties. X and Y are called *stably birational* if there's n, m such that $X \times \mathbb{P}^n$ is birational to $Y \times \mathbb{P}^m$.

Theorem ([LL03])**.** *X and Y are stably birational if and only if $e(X) \cong e(Y) \bmod (\mathbb{L})$ in $K_0(\mathrm{Var}_k)$.*

Proof of the "only if" claim using weak factorization. If $X \times \mathbb{P}^n$ is birational to $Y \times \mathbb{P}^m$, then it suffices to show that $e(X \times \mathbb{P}^n) = e(Y \times \mathbb{P}^m)$ modulo $\mathbb{L}$, since $e(X \times \mathbb{P}^n) = e(X)e(\mathbb{P}^n)$, which is just $e(X)$ modulo $\mathbb{L}$. Thus we may assume X and Y are birational.

Using the weak factorization theorem of [AKMW02] above, the birational equivalence between X and Y can be expressed via chain of blowing up and blowing down along smooth centers, so we can reduce to the case of a single blowup $Y = \mathrm{Bl}_Z X \to X$. In this case though, the result is immediate: we've seen above that

$$e(Y) - e(E) = e(X) - e(Z),$$

and that also

$$e(E) = e(Z)(\mathbb{L}^{c-1} + \cdots + \mathbb{L} + 1).$$

Thus, modulo $\mathbb{L}$ we have immediately that

$$e(Y) - e(Z) = e(X) - e(Z),$$

so $e(Y) = e(X)$ modulo $\mathbb{L}$, as desired. $\square$

This has been used, for example, along with techniques inspired by motivic integration, to prove the following theorem, which says essentially that stable birationality specializes in families with mild singularities:

Theorem ([NS19]). *Let $f : \mathcal{X} \to C$ be a proper flat morphism with C a smooth connected curve. If a very general (closed) fiber of f is stably birational, so is any (closed) fiber with at worst ordinary double points.*

3 Arc schemes

We've constructed the "target" of motivic integration (the ring $\widehat{\mathcal{M}}_k$); now we need to define where our "measurable" sets will live. This is the *arc scheme* of a k-variety. Why is this a natural concept to consider? Recall that the idea in Batyrev's proof was to calculate point counts over a finite field by a p-adic integral, i.e., to consider the $\mathbb{Z}_p$-valued points of a scheme (or really the $\mathbb{Q}_p$-points), i.e., morphisms $\operatorname{Spec}\mathbb{Z}_p \to X$. Working with varieties over some field k, we'll replace $\mathbb{Z}_p$ by a more natural DVR over k, the ring $k[[t]]$. The goal is thus to build a space parametrizing morphisms $\operatorname{Spec} k[[t]] \to X$ for a given k-variety X; the points of this space (i.e., $\operatorname{Spec} k[[t]]$-points of X) will then be the ambient space on which we'll do integration.

Remark. It may be easier to think of this description in the affine local picture: if $X = \operatorname{Spec} R$, a morphism $\operatorname{Spec} k[[t]] \to \operatorname{Spec} R$ is the same thing as a ring map $R \to k[[t]]$ (and likewise for morphisms $\operatorname{Spec} k[t]/t^{\ell+1}$). If $R = k[x_1, \ldots, x_n]$, then this is the same thing as giving power series

$$x_1(t) = \sum x_1^{(j)} t^j, \ldots x_n(t) = \sum x_n^{(j)} t^j,$$

which is the exact same thing as giving the coefficients $x_i^{(j)}$ of the power series. Thus, we can parametrize these morphisms by "infinite affine space", corresponding to the coefficients of the power series.

In order to construct a parameter space as a scheme though, we have to think not only of the "closed points" of the space of arcs, but of its scheme structure as well, so we now proceed with the formal description.

Since $k[[t]]$ is the inverse limit of the rings $k[t]/t^{\ell+1}$, to parametrize morphisms $\operatorname{Spec} k[[t]] \to X$ we first examine how to parametrize morphisms $\operatorname{Spec} k[t]/t^{\ell+1} \to X$.

Let k be a field and let X be an essentially finite-type k-scheme. For each $\ell \in \mathbb{N}$ consider the functor

$$T \mapsto \operatorname{Hom}_k(T \times_k \operatorname{Spec}(k[t]/t^{\ell+1}), X)$$

from k-schemes to sets. A k-morphism $T \times_k \operatorname{Spec} k[t]/t^{\ell+1} \to X$ is called an ℓ-jet on X.

Theorem. *This functor is representable, i.e., there exists a scheme X_ℓ such that*

$$\operatorname{Hom}(T, X_\ell) \cong \operatorname{Hom}(T \times_k \operatorname{Spec}(k[t]/t^{\ell+1}), X)$$

are naturally isomorphic functors.

Remark. Note that for $\ell = 1$ this is the same as the functor representing the tangent space of X, so that the representing scheme is just the total tangent space of X.

We call the representing k-scheme X_ℓ, the ℓ-th jet scheme of X. Note that the proof demonstrates that if X is finite-type over k, so is each X_ℓ. By definition the closed points of X_ℓ corresponds to an ℓ-jet $\operatorname{Spec} k[t]/t^{\ell+1} \to X$, which we think of as an "infinitesimal part of a curve" on X.

Example. We present the proof only in the case where $X = \operatorname{Spec} R$, with R a finitely generated algebra over k (we'll shortly see that behaves well under localization, so the general case isn't so far off). Translating everything into the language of rings, we want to construct a k-algebra R_ℓ such that for k-algebra C, giving a k-algebra morphism $R_\ell \to C$ is the same thing as giving a k-algebra morphism $R \to C[t]/t^{\ell+1}$.

Say $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$. Giving a morphism

$$R \to C[t]/t^{\ell+1}$$

is the same as giving a map

$$k[x_1, \dots, x_n] \to C[t]/t^{\ell+1}$$

that is zero on $(f_1, \dots, f_m)$. Giving a map $k[x_1, \dots, x_n] \to C[t]/t^{\ell+1}$ is the same as giving n elements

$$x_i(t) := \sum_{i=0}^{\ell} x_i^{(j)} t^j \in C[t]/t^{\ell+1}$$

(i.e., each $x_i^{(j)}$ is an element of C). We have that this factors through R itself exactly when each

$$f_s\Big(x_1(t) \dots, x_n(t)\Big) = 0$$

in $C[t]/t^{\ell+1}$. But this is a condition only on the coefficients of each t^i, so we can expand to

$$f_s^{(0)}(x_1(t), \dots, x_n(t)) + f_s^{(1)}(x_1(t), \dots, x_n(t))t + \dots + f_s^{(\ell)}(x_1(t), \dots, x_n(t))t^{\ell}.$$

The condition then is that each $f_s^{(i)}(x_1(t), \dots, x_n(t)) = 0$.

So, giving a map $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m) \to C[t]/t^{\ell+1}$ is the same as giving elements

$$x_i^{(j)}, \quad i = 1, \dots, m, j = 1, \dots, \ell$$

of C satisfying the equations

$$f_s^{(t)}(x_1(t), \dots, x_n(t)) = 0, \quad s = 1, \dots, m, t = 1, \dots, \ell.$$

This is the same as giving a morphism from

$$k\big[x_i^{(j)}\big]/\big(f_s^{(t)}(x_1(t), \dots, x_n(t))\big) \to C,$$

so that we can just take

$$R_\ell := k\big[x_1^{(j)}, \dots, x_n^{(j)} : j = 0, \dots, \ell\big]/\big(f_s^{(t)}(x_1(t), \dots, x_n(t))\big).$$

The takeaway here is that constructing these rings is straightforward but the number of variables and equations involved rapidly grow. Note also that if $Z \subset X$ is a closed inclusion then we have a natural closed inclusion $Z_\ell \subset X_\ell$.

Example. (1) Consider $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$. Then $X_\ell = \operatorname{Spec} k[x_0, \dots, x_\ell, y_0, \dots, y_\ell] \cong \mathbb{A}^{2(\ell+1)}$.

(2) Consider $X = \operatorname{Spec} k[x, y]/(x^2 + y^3)$. Say we want to find X_2. From the above approach, we will construct X_2 as a closed subset of

$$\mathbb{A}_2^2 = \operatorname{Spec} k[x_0, x_1, x_2, y_0, y_1, y_2].$$

A point $(x_0, x_1, x_2, y_0, y_1, y_2) \in \mathbb{A}_2^2$, though of as the 2-jet

$$\big(x_0 + x_1 t + x_2 t^2, y_0 + y_1 t + y_2 t^2\big)$$

will lie in X_2 exactly when

$$(x_0 + x_1 t + x_2 t^2)^2 + (y_0 + y_1 t + y_2 t^2)^3.$$

We multiply out and gather powers of t to obtain

$$(x_0^2 + y_0^3) + (2x_0 x_1 + 3y_0^2 y_1)t + (x_1^2 + 2x_0 x_2 + 3y_0 y_1^2 + 3y_0^2 y_2)t^2.$$

Thus, we have that

$$X_2 = \operatorname{Spec} \frac{k[x_0, x_1, x_2, y_0, y_1, y_2]}{(x_0^2 + y_0^3, 2x_0 x_1 + 3y_0^2 y_1, x_1^2 + 2x_0 x_2 + 3y_0 y_1^2 + 3y_0^2 y_2)}.$$

Remark. One can check that X_2 is neither reduced nor irreducible, thus suggesting that jet schemes of singular varieties become bad quite quickly.

Remark. We can also (in characteristic 0 at least) obtain the defining equations of the jet schemes via "formal differentiation". This is most clearly seen through an example: Starting with $X = \operatorname{Spec} k[x,y]/x^2+y^3$, we think of x as x_0 and y as y_0. We then "differentiate" $x_0^2+y_0^3$, obtaining $2x_0x_1+3y_0^2y_1$,which is the second of the defining equations for X_2 above. Differentiating one more time, we get $2x_1^2+2x_0x_2+6y_0y_1^2+3y_0^2y_2$, which is the two times the third defining equation minus the second.

Exercise. Adapt this to positive characteristic.

Remark. Note that there is a natural map $X_\ell \to X$: a T-point of X_ℓ determines an arc

$$T \times \operatorname{Spec} k[t]/t^{\ell+1} \to X,$$

and we simply precompose with the map $T = T \times \operatorname{Spec} k \to T \times \operatorname{Spec} k[t]/t^{\ell+1}$ corresponding to killing t. In more down-to-earth terms, this just sends a jet to the closed point its based at.

The truncation maps $k[t]/t^{\ell+1} \to k[t]/t^{\ell'+1}$ for $\ell' < \ell$ induce morphisms $\psi^X_{\ell,\ell'} : X_\ell \to X_{\ell'}$, which are easily checked to be affine, so we obtain an inverse system $\{\cdots \to X_\ell \to X_{\ell-1} \to \cdots\}$ of affine morphisms. We can thus form the inverse limit, which we denote by X_∞ and call the arc scheme of X. In contrast to the X_ℓ, this is almost never finitely generated. We have truncation maps $\psi^X_\ell := \psi^X_{\infty,\ell} : X_\infty \to X_\ell$ for all ℓ.

Remark. From what we've just seen, these truncation morphisms are surjective when X is smooth; in fact, this will be equivalent to X being smooth, and in general if X is not smooth they are quite far from surjective.

Again, the functor $X \to X_\infty$ is functorial, so a morphism of schemes $f : X \to Y$ induces a morphism $f_\infty : X_\infty \to Y_\infty$.

Remark. X_∞ is also straightforward to construct from X; one works on affine pieces $\operatorname{Spec} R$, where we choose some presentation $R = k[x_1,\ldots,x_n]/(f_1,\ldots,f_m)$. Giving a map $R \to C[[t]]$ corresponds to giving n power series

$$x_i(t) := \sum_{i=0}^{\infty} x_i^{(j)} t^j \in C[[t]].$$

such that each $f_s(x_1(t),\ldots,x_n(t))$ is zero. Thus, again we just expand this out and rewrite each $f_s(x_1(t),\ldots,x_n(t))$ as

$$f_s^{(0)}(x_1(t),\ldots,x_n(t)) + f_s^{(1)}(x_1(t),\ldots,x_n(t))t + \ldots.$$

Then we just take $R_\infty = k[x_1^{(j)},\ldots,x_n^{(j)} : j = 0,1,2,\ldots]/(f_1^{(i)},\ldots,f_s^{(i)},\cdots : i = 0,1,\ldots)$. Note that the defining ideal is very far from finitely presented!

Remark. One can check that if $k \hookrightarrow L$ is a field extension then

$$\operatorname{Hom}(\operatorname{Spec}(L[[t]]), X) = \operatorname{Hom}(\operatorname{Spec}(L), J_\infty(X)).$$

In fact, by a theorem of Bhargav it is true (but highly nonelementary) that if X is quasicompact and quasiseparated over k and S is a k-algebra then

$$\operatorname{Hom}\big(\operatorname{Spec} S \times_k \operatorname{Spec}(k[[t]]/t^{\ell+1}), X\big) = \operatorname{Hom}(\operatorname{Spec} S, J_\ell(X)),$$

but we do not use this in the following.

Remark. Note that given any morphism $f : X \to Y$ we obtain a morphism $f_\ell : X_\ell \to Y_\ell$, defined on the level of functor of points by sending a morphism

$$T \times \operatorname{Spec} k[t]/t^{\ell+1} \to X$$

to the composition

$$T \times \operatorname{Spec} k[t]/t^{\ell+1} \to Y.$$

These maps then induce a map on arc schemes

$$f_\infty : X_\infty \to Y_\infty.$$

Proposition. *If $f : X \to Y$ is an étale morphism of k-schemes, we have a pullback square*

$$\begin{tikzcd} X_\infty = Y_\infty \times_Y X \arrow[r] \arrow[d] & Y_\infty \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

and likewise for the ℓ-jet schemes.

Exercise. Prove this (hint: it's enough to use that f is formally étale and work with the ℓ-jet schemes).

Corollary. *Formation of the jet schemes commutes with open immersions.*

Corollary. *If X is smooth of dimension n then X_ℓ is an $n(\ell+1)$-bundle over X (and in particular is also smooth).*

Proof. It suffices to work on some affine open $U \subset X$; since U is smooth of dimension n there is an étale morphism $U \to \mathbb{A}^n$, and thus by the proposition it suffices to work in the case of $\mathbb{A}^n$, where by our construction of the jet schemes this is immediate. ☐

Cylinders in the space of arcs

Now, we want to give a class of certain "nice" sets in the space of arcs, which will underlie our construction of measurable sets. These will be the cylinders in the space of arcs.

Definition. Let X be a variety. A cylinder in X_∞ is a subset of the form

$$(\psi_\ell^X)^{-1}(S)$$

for S a constructible subset of X_ℓ (recall that a constructible subset is the union of locally closed sets).

Intuitively, a cylinder should be thought of as a set in X_∞ in which membership can be verified by looking at whether finitely many coefficients are zero or nonzero; for example, if $Z \subset X$ is a proper closed subset, Z_∞ will never be a cylinder.

One can check that the intersection and union of cylinders are still cylinders, which we'll use frequently in the following.

Note that cylinders could be open or closed in the topology of X_∞; for example, if $X = \operatorname{Spec} k[x, y]$, then we've seen that $X_\infty = \operatorname{Spec} k[x_0, x_1, \dots, y_0, y_1, \dots]$, and:

(1) $V(y_0)$, the set of arcs passing through the x-axis, is a closed cylinder.

(2) $D(y_0)$, the set of arcs *not* passing through the x-axis, is an open cylinder.

(3) $V(y_0, y_1, \dots,)$, the set of arcs contained in the x-axis, is not a cylinder.

There is a natural type of cylinder we'll consider often, which are the contact loci:

Definition. First, we note that to any arc $\gamma : \operatorname{Spec} k[[t]] \to X$, say factoring through Z, we can associate a semivaluation $\operatorname{ord}_\gamma(-)$ on $k(X)$, given by sending $f \in \mathcal{O}_X$ to $\operatorname{ord}_t \gamma^*(f)$ (i.e., pull back f along the map $\gamma : \operatorname{Spec} k[[t]] \to X$ and apply the t-adic valuation on $k[[t]]$). We define also $\operatorname{ord}_\gamma(I) := \min(\operatorname{ord}_t(f) : f \in I)$.

Let $Z \subset X$ be a closed subscheme (not necessarily reduced!), defined by an ideal sheaf I_Z. We define

$$\mathrm{Cont}^{\geq e}(Z) = \{\gamma \in X_\infty : \mathrm{ord}_t\, \gamma^*(I_Z) \geq e\}.$$

This is a (closed) cylinder, because clearly

$$\mathrm{Cont}^{\geq e}(Z) = \psi_{e+1}^{-1}(Z_{e+1}).$$

Then

$$\mathrm{Cont}^{e}(Z) = \{\gamma \in X_\infty : \mathrm{ord}_t\, \gamma^*(I_Z) = e\} = \mathrm{Cont}^{\geq e}(Z) - \mathrm{Cont}^{\geq e+1}(Z)$$

is a (locally closed) cylinder.

Motivic measures of cylinders

We now specialize to the case where $\mathrm{char}\, k = 0$, and moreover where X is smooth, although the theory can be developed in the singular case as well.

Remark. When X is smooth, the morphisms $\psi_{\ell,\ell'} : X_\ell \to X_{\ell'}$ will be surjective: since formation of jet schemes commutes with étale basechange, it suffices to check this for $X = \mathbb{A}^n$; in this case, one can easily check that the map $(\mathbb{A}^n)_\ell \to (\mathbb{A}^n)_{\ell'}$ is induced by the inclusion of rings

$$k[x_i^{(j)} : i = 1, \dots, n,\, j = 0, \dots, \ell'] \hookrightarrow k[x_i^{(j)} : i = 1, \dots, n,\, j = 0, \dots, \ell];$$

this is a faithfully flat ring extension (it's just adjoining some more indeterminates!) and thus the map on Spec is surjective. In fact, we see from this description that $X_\ell \to X_{\ell'}$ is actually an $\mathbb{A}^{n(\ell-\ell')}$-bundle (again, this is clear in the case where $X = \mathbb{A}^n$, and basechanging a bundle gives another bundle).

From this, one can check that the truncation maps $\psi_\ell : X_\infty \to X_\ell$ are also surjective.

This in fact characterizes smoothness.

Remark. This has the following consequence for cylinders in the space of arcs: Say $\dim X = n$. Say $C = \psi_{\ell_0}^{-1}(S_0)$ for $S_0 \subset X_{\ell_0}$ constructible. By commutativity of the diagram

$$\begin{tikzcd}
X_\infty \arrow[dd, "\psi_{\ell_0}"'] \arrow[dr, "\psi_\ell"] & \\
& X_\ell \arrow[dl, "\psi_{\ell,\ell_0}"] \\
X_{\ell_0} &
\end{tikzcd}$$

we have that $C = \psi_\ell^{-1}\big(\psi_{\ell,\ell_0}^{-1}(S_0)\big)$ as well. Set $S = \psi_{\ell,\ell_0}^{-1}(S_0)$. We want to analyze the relation between S and S_0, in order to get an invariant of C that doesn't depend on what level we write it as being pulled back from.

From our preceding remark, since X is smooth(!), $X_\ell \to X_{\ell_0}$ is an $\mathbb{A}^{(\ell-\ell_0)n}$-bundle, so we see that S is an $\mathbb{A}^{(\ell-\ell_0)n}$-bundle over S_0.

Now, let's consider this relation in the Grothendieck ring of varieties: Recalling that the Grothendieck ring "trivializes bundles", we have that

$$e(S) = e(S_0)e(\mathbb{A}^{(\ell-\ell_0)n}) = e(S_0)\mathbb{L}^{(\ell-\ell_0)n}$$

in $K_0(\mathrm{Var}_{\mathbb{C}})$.

Now, we see the first hint of the necessity of inverting $\mathbb{L}$: in $\mathcal{M}_C$, we can multiply the above expression by $\mathbb{L}^{-\ell n}$ to get that

$$e(S)\mathbb{L}^{-n\ell} = e(S_0)\mathbb{L}^{-n\ell_0}.$$

From this calculation, we obtain the following:

Definition. Let X be a smooth variety, and say $C \subset X_\infty$ is a cylinder. Then we can write $C = \psi_\ell^{-1}(S)$ for $S \subset X_\ell$ a constructible subset; we define the motivic measure of C to be

$$\mu(C) = e(S)\mathbb{L}^{-n\ell}.$$

From what we've just seen, this expression is independent of the choice of X_ℓ which we view C as being pulled back from. This is the necessity of inverting $\mathbb{L}$ in the Grothendieck ring: it allows us to give a stable meaning to certain nice subsets of non-finite-type schemes.

Remark. By surjectivity of the truncation maps $\psi_\ell : X_\infty \to X_\ell$, we have that $\psi_\ell(\psi_\ell^{-1}(S)) = S$; thus, we can just as well define $\mu(C)$ to be the class $e(\psi_\ell(S))\mathbb{L}^{-n\ell}$ for $\ell \gg 0$.

Example. Note that X_∞ is a cylinder; it's just $\psi_0^{-1}(X)$. As such, we see that

$$\mu(X_\infty) = e(X).$$

Thus, the class of X in the Grothendieck group has been rephrased as a certain "motivic measure" of a subset of the arc space (in fact, the entire arc space).

Measurable sets

The cylinders in the space of arcs will be the "basic" sets we'll use to approximate the measure of more complicated sets, just as balls or n-cubes in $\mathbb{R}^n$ are the sets we use to approximate ordinary integrals. This approximation will make clear the need to complete $\mathcal{M}_\mathbb{C}$ to obtain $\widehat{\mathcal{M}}_\mathbb{C}$.

Remark. In fact, for our purposes, the main kind of "noncylinder" set we want to assign a measure to is $Z_\infty \subset X_\infty$, where $Z \subset X$; we've previously referred to these as "thin" sets, and we want to give them measure zero in a rigorous sense.

Definition. A subset $C \subset X_\infty$ is called measurable if for every N there is a cylinder C_N and countably many cylinders $D_{N,i}$ such that

$$(C - C_N) \cup (C_N - C) \subset \bigcup D_{N,i},$$

with $\mu(D_{N,i}) \in F^N\mathcal{M}_\mathbb{C}$. In this case, we define the motivic volume of C is defined to be

$$\mu(C) := \lim_{N\to\infty} \mu(C_N) \in \widehat{\mathcal{M}}_\mathbb{C}.$$

Remark. This should be thought of as completely analogous to usual measure theory: we're approximating C by a cylinder C_N (an "elementary measurable set"), such that the difference between C and C_N is contained in a countable union of "small" cylinders, i.e., cylinders whose motivic volumes are in a high-degree piece of the filtration on $\mathcal{M}_\mathbb{C}$.

Remark. Recall also that the filtration on $\mathcal{M}_\mathbb{C}$ is really just the filtration by dimension, where $\dim(e(S\mathbb{L}^{-d})) = \dim(S) - d$.

We won't justify the assertions of the definition in detail, but we will mention the following crucial input:

Lemma. *Let X be smooth over an uncountable field k, and let $C_1 \supset C_2 \supset \dots$ be a decreasing sequence of nonempty cylinders in X_∞. The intersection $\bigcap C_i$ is nonempty.*

This in turn depends upon the corresponding statement that a decreasing sequence of constructible subsets of an algebraic variety over an uncountable field has nonempty intersection; from this, one constructs a sequence of ℓ-jets γ_ℓ lying in the image of C_ℓ, and thus an arc γ in each C_i.

Definition. A function $\alpha : X_\infty \to \mathbb{N} \cup \{\infty\}$ is called measurable if the level sets $\alpha^{-1}(m)$ are measurable for each $m \in \mathbb{N} \cup \{\infty\}$ and $a^{-1}(\infty)$ has measure 0. If α is measurable, we define the motivic integral of α as

$$\int_{X_\infty} \mathbb{L}^{-\alpha}\, d\mu := \sum_{m\in\mathbb{N}\cup\{\infty\}} \mu(\alpha^{-1}(m)) \cdot \mathbb{L}^{-m} \in \hat{\mathcal{M}}_\mathbb{C}.$$

Remark. Note that we do not need to impose any convergence condition on the right side, once we know that $\alpha^{-1}(\infty)$ has measure 0. Since $\widehat{\mathcal{M}}_{\mathbb{C}}$ is a complete ring, with topology arising from the filtration on $\mathcal{M}_C$, existence of the right-hand side is the same as demanding that

$$\mu(\alpha^{-1}(m))\mathbb{L}^{-m}$$

converges to 0 in $\widehat{\mathcal{M}}_C$ as $m \to \infty$. When $\alpha^{-1}(m)$ is an actual cylinder for finite m (which will happen in all the cases we consider) this is the same as demanding that $\mu(\alpha^{-1}(m))\mathbb{L}^{-m}$ lie in $F^{d_m}\mathcal{M}_{\mathbb{C}}$ for $d_m \to \infty$, which just says that

$$\dim\bigl(\mu(\alpha^{-1}(m))\mathbb{L}^{-m}\bigr) = \dim\bigl(\mu(\alpha^{-1}(m))\bigr) - m$$

converges to $-\infty$ when m goes to ∞. But note that clearly $\dim\bigl(\mu(\alpha^{-1}(m))\bigr) \leq \dim \mu(X_\infty) = \dim X$, and so

$$\dim\bigl(\mu(\alpha^{-1}(m))\bigr) - m \leq \dim X - m,$$

and the right side clearly goes to $-\infty$ as m goes to ∞.

Remark. The main example of measurable functions, and the only example we'll use in this course, arises in the following way: let $D \subset X$ be an effective divisor (so locally D corresponds to the data of a defining equation), and take $\alpha(\gamma) = \mathrm{ord}_\gamma(D)$. We'll write this as $\alpha = \mathrm{ord}(D)$. The level sets $\alpha^{-1}(m)$ for $m < \infty$ then are just the contact loci

$$\mathrm{Cont}^m(D);$$

thus, they are all immediately seen to be measurable. The only thing left in order for $\mathrm{ord}(D)$ to be measurable is that $\mathrm{ord}_\infty(D) = D_\infty \subset X_\infty$ should be measurable and have measure 0. We'll postpone this check briefly.

Proposition. *μ is additive on finite disjoint union; moreover, if C_i is a disjoint sequence of measurable sets with $\mu(C_i) \to 0$, then $C = \bigcup C_i$ is measurable and $\mu(C) = \sum \mu(C_i)$.*

The birational transformation formula

We now state the key theorem in applications of arc spaces to birational geometry. From now on, when we reference X_∞, X_ℓ, or X, we refer to their k-points only.

The following is one of the key reasons for the relevance of arc spaces to the study of birational geometry:

Theorem. *Let $f : X \to Y$ be a proper birational morphism, and say f is an isomorphism away from a closed subset $Z \subset Y$. Then*

$$f_\infty : X_\infty - (f^{-1}(Z))_\infty \to Y_\infty - Z_\infty$$

is a bijection of sets.

That is, away from the "thin" sets Z_∞ and $f^{-1}(Z)_\infty$, the map is a bijection. Note that Z_∞ consists exactly of the arcs $\mathrm{Spec}\, k[[t]] \to X$ that factor through the inclusion $Z \hookrightarrow X$.

Remark. Note that Z_∞ is *not* the arcs with closed point in Z, but rather the arcs lying entirely in Z! For example, if $Y = \mathbb{A}^2$ and $Z = V(y)$ is the x-axis, we have that $Y_\infty = \mathrm{Spec}\, k[x_0, x_1, \ldots, y_0, y_1, \ldots]$ and $Z_\infty = V(y_0, y_1, \ldots)$. That is, an arc passes through the x-axis if it lies in the codimension-1 subset $V(y_0)$, but is in Z_∞ if it lies in the infinite-codimension subset $V(y_0, y_1, \ldots)$. The difference between sets with finite and infinite codimensions is at the heart of the theory of arc spaces.

Proof. Let $\gamma \in Y_\infty - Z_\infty$. We can view γ as a map $\mathrm{Spec}\, k[[t]] \to Y$; moreover, since the generic point $\eta := \mathrm{Spec}\, k((t)) \hookrightarrow \mathrm{Spec}\, k[[t]] \to Y$ does not lie in Z, f is an isomorphism over η, so we can lift it to a "punctured arc" $\mathrm{Spec}\, k((t)) \to X$. We thus obtain the diagram

$$\begin{tikzcd}
\mathrm{Spec}\, k((t)) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\mathrm{Spec}\, k[[t]] \arrow[r, "\gamma"] \arrow[ur, dashed, "\tilde{\gamma}"] & Y
\end{tikzcd}$$

Since $k[[t]]$ is a DVR, the valuative criteria for properness gives immediately that there exists a unique $\widetilde{\gamma}$ making the diagram commute. □

Thus, X_∞ and Y_∞ should be thought of as quite closely related; this is made precise through the following theorem:

Remark. The birational transformation rule requires the notion of the relative canonical divisor $K_{Y/X} = K_Y - f^*K_X$ given a morphism $f : Y \to X$. In our setting, we will need to define this only when f is proper and birational and Y, X are smooth of the same dimension. In this case, we can define $K_{Y/X}$ in a familiar way: locally on Y, choose an algebraic system of coordinates $y_1, \ldots, y_n$, and say the morphism f is given by n polynomial functions $f_1, \ldots, f_n$ in the y_i. Then we form the square matrix of partial derivatives

$$\left(\partial f_i / \partial x_j\right);$$

$K_{Y/X}$ is then just the codimension-1 subvariety of Y cut out by the determinant of this matrix.

The following theorem, due to Kontsevich in the smooth case and Denef and Loeser [DL99] in the singular case, will allow us to connect information about $K_{Y/X}$ to the arc spaces; in particular, it connects the behavior of the induced morphisms of jet schemes to the order of vanishing along $K_{Y/X}$.

Theorem (birational transformation rule)**.** *Let $f : Y \to X$ be a proper birational morphisms of smooth complex varieties and let α be a measurable function on X_∞. Then*

$$\int_{X_\infty} \mathbb{L}^{-\alpha}\, d\mu_X = \int_{Y_\infty} \mathbb{L}^{-(\alpha \circ f_\infty) - \operatorname{ord}(K_{Y/X})}\, d\mu_X.$$

Theorem (jet-theoretic version)**.** *Let $f : X' \to X$ be a proper birational morphism with X and X' nonsingular. For e write*

$$C_e = \operatorname{Cont}^e(K_{X/Y}).$$

Then for $m \geq \max(2e)$ the map on m-jets

$$\psi_m^{X'}(C_e) \to f_m(\psi_m^{X'}(C_e))$$

is an $\mathbb{A}^e$-fibration.

By a short argument, this implies that the image of each $C'_{e,m}$ is a cylinder.

Note that this describes what happens away from the thin set $(K_{Y/X})_\infty$.

Proof of the birational transformation rule via the statement on jets. Say f is an isomorphism away from $Z \subset X$ (so $K_{Y/X}$ is supported on $f^{-1}(Z)$). Let $C'_e = \operatorname{Cont}^e(K_{Y/X})$, and set

$$C'_{e,m} = (\alpha \circ f_\infty)^{-1} \cap C'_e.$$

Let $C_{e,m} = f_\infty(C'_{e,m})$; the fact that this is a cylinder is a consequence of the jet-theoretic statement.

It's then clear that

$$X_\infty = \sqcup C_{e,m}, \quad Y_\infty = \sqcup C'_{e,m},$$

at least away from the sets Z_∞ and $(f^{-1}(Z))_\infty = (K_{Y/X})_\infty$; these have measure 0 in any case, so we can ignore them in our integration. From the statement on jets, we have that $\mu(C'_{e,m}) = \mu(C_{e,m})\mathbb{L}^e$. Then, we have that

$$\int_{X_\infty} \mathbb{L}^{-\alpha} = \sum_m \mu(\alpha^{-1}(m))\mathbb{L}^{-m} = \sum_{m,e} \mu(C_{e,m})\mathbb{L}^{-m} = \sum_{m,e} \mu(C'_{e,m})\mathbb{L}^{-m-e} = \sum_{m,e} \mu(C'_{e,m})\mathbb{L}^{-m-e};$$

finally, note that if we collect the right side by the values of $-m-e$, we get exactly the level sets of $-(\alpha \circ f_\infty + \operatorname{ord}(K_{Y/X}))$, so the right side is just $\int_{Y_\infty} \mathbb{L}^{-(\alpha \circ f_\infty + \operatorname{ord}(K_{Y/X}))}$. □

This statement should be contrasted with what happens on the level of the arc scheme: the map is a *bijection* there, but on the finite-level jet schemes it is a fibration of varying dimension! Thus, one can think of the map on the arc schemes as being a bijection that changes the codimension of various cylinders.

"Proof" of the birational transformation rule. Following [Bli11], we give a proof only in the case of a blowup of a smooth variety along a smooth subvariety. We will then appeal to the weak factorization theorem to claim the result for general proper birational morphisms of smooth projective varieties. The original proof is in fact much more elementary and direct, and works for arbitrary birational morphisms without the use of the weak factorization theorem, but gives less intuition for what's going on.

We may work locally, and moreover since formation of jet schemes commutes with étale morphisms, starting from a smooth variety X we may assume that $X \cong \mathbb{A}^n$, say with coordinates $x_1, \dots, x_n$, with $Z \subset X$ the closed subvariety cut out by $x_1, \dots, x_c$. The blowup $\mathrm{Bl}_Z X$ is then covered by charts of the form

$$\operatorname{Spec} k\left[\frac{x_1}{x_i}, \dots, x_i, \dots, \frac{x_c}{x_i}, x_{c+1}, \dots, x_n\right]$$

Without loss of generality, let $i = c$. The blowup map corresponds to the ring inclusion

$$k[x_1, \dots, x_n] \hookrightarrow k\Big[x_1/x_c, \dots, x_{c-1}/x_c, x_c, x_{c+1}, \dots, x_n\Big].$$

Relabeling the variables on the left side $y_1, \dots, y_n$, this is the ring map

$$k[x_1, \dots, x_n] \to k[y_1, \dots, y_n],$$

sending

$$\begin{cases} x_1 \mapsto y_1 y_c \\ x_2 \mapsto y_1 y_2 \\ \vdots \\ x_{c-1} \mapsto y_{c-1} y_c \\ x_c \mapsto y_c \\ x_{c+j} \mapsto y_{c+j} \end{cases}$$

First, we calculate $K_{Y/X}$: the Jacobian matrix of the map is just

$$\begin{pmatrix} y_c & & \cdots & & & & & \\ & y_c & & & & & & \\ & & \ddots & & & & & \\ \vdots & & & y_c & & & & \\ y_1 & y_2 & \cdots & y_{c-1} & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

The determinant of this matrix is clearly just y_c^{c-1}, so that $K_{Y/X} = (c-1)E$.

Consider an ℓ-jet on Y with contact order e with $K_{Y/X}$; note that we must have that $c - 1 \mid e$ for this to happen. Say that $e = (c-1)e_0$. Choose $\ell \gg 0$ (or just $\geq 2e$). An ℓ-jet of contact order e_0 along y_c has the form

$$\begin{cases} y_i(t) = \sum_{i=0}^{\ell} y_i^{(j)} t^j & i \neq c, \\ y_c(t) = y_c^{(e_0)} t^{e_0} + \cdots + y_c^{(\ell)} t^{\ell} & y_c^{(e_0)} \neq 0. \end{cases}$$

This ℓ-jet is mapped to the ℓ-jet

$$\begin{cases} x_1(t) = y_1(t)y_c(t) \\ \vdots \\ x_{c-1}(t) = y_{c-1}(t)y_c(t) \\ x_c(t) = y_c(t) \\ \vdots \\ x_n(t) = y_n(t) \end{cases}$$

on X.

Now, we need to ask what the fiber over general point in this image is. Examining the above equations, one can see that a general point in the image is a power series $x_1(t), \dots, x_n(t)$ such that $x_1, \dots, x_{c-1}$ are of the form $t^{e_0}x_1'(t), \dots, t^{e_0}x_{c-1}'(t)$.

First, note that $y_c(t), \dots, y_n(t)$ are specified uniquely by $x_c(t), \dots, x_n(t)$. For each $x_i(t)$ for $i = 1, \dots, c-1$, we proceed as follows: we can write

$$y_c(t) = t^{e_0}(y_c^{(e_0)} + \cdots + y_c^{(\ell)}t^{\ell-e_0});$$

since $y_c^{(e_0)}$ is invertible, the quantity $y_c'(t) := y_c^{(e_0)} + \cdots + y_c^{(\ell)}t^{\ell-e_0}$ is a unit, and thus the equality

$$x_i(t) = y_1(t)y_c(t)$$

can be rewritten as

$$x_i(t)/y_c'(t) = t^{e_0}y_i(t).$$

The left side is just some truncated power series in t, *which depends only on the* $x_i(t)$ whose fiber we're considering. The above equality is then the same as

$$z_{e_0}t^{e_0} + \cdots + z_\ell t^\ell = t^{e_0}y_i(t).$$

for some fixed coefficients z_i (where we use that $x_i(t)$ is divisible by t^{e_0}). Finally, expand out the right side, to obtain

$$z_{e_0}t^{e_0} + \cdots + z_\ell t^\ell = y_i^{(0)}t^{e_0} + y_i^{(1)}t^{e_0+1} + \cdots + y_i^{(\ell-e_0)}t^\ell.$$

Thus, we see that $y_i^{(j)}$ is uniquely fixed for $j = 0, \dots, \ell - e_0$, while $y_i^{(j)}$ is completely arbitrary for $\ell - e_0 + 1 \le j \le \ell$. Thus, we may freely choose e_0 parameters for each of the $c-1$ power series $y_1(t), \dots, y_{c-1}(t)$, so the fiber is just $\mathbb{A}^{(c-1)e_0} = \mathbb{A}^e$. $\square$

Remark. Finally, we want to sketch why for our purposes it suffices to treat the case of a smooth blowup along a smooth center. First, we note that this gives the result for the composition of such blowups: If the birational transformation theorem is true for $f : Y \to X$ and $g : Z \to Y$, then we have

$$\int_{X_\infty} \mathbb{L}^{-\alpha} \int_{Y_\infty} \mathbb{L}^{-(\alpha\circ f_\infty)-\mathrm{ord}(K_{Y/X})} \int_{Z_\infty} \mathbb{L}^{-(\alpha\circ f_\infty\circ g_\infty)-\mathrm{ord}(g^*K_{Y/X})-\mathrm{ord}(K_{Z/Y})};$$

one then uses that

$$\mathrm{ord}(g^*K_{Y/X}) + \mathrm{ord}(K_{Z/Y}) = \mathrm{ord}(g^*K_{Y/X} + K_{Z/Y})$$

and

$$g^*K_{Y/X} + K_{Z/Y} = K_Z - g^*K_Y + g^*K_Y - (g\circ f)^*K_X = K_Z - (g\circ f)^*K_X = K_{Z/X}.$$

Now, we note that if we have a diagram

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X_1 \arrow[rr, "g"] & & X_2 \end{tikzcd}$$

and the theorem holds for f_1 and f_2, then it holds for g:

Since

$$K_{Y/X_2} = K_{Y/X_1} + f_1^* K_{X_1} - f_2^* K_{X_2},$$

and

$$K_{X_1} = K_{X_1/X_2} - g^* K_{X_2},$$

we have

$$f_1^* K_{X_1} = f_1^* K_{X_1/X_2} - f_1^*(g^* K_{X_2}) = f_1^* K_{X_1/X_2} - f_2^* K_{X_2},$$

so that

$$K_{Y/X_2} = K_{Y/X_1} + f_1^* K_{X_1/X_2}.$$

Thus, if α is a measurable function on $(X_2)_\infty$,

$$\begin{aligned}\int_{(X_2)_\infty} \mathbb{L}^{-\alpha} &= \int_{Y_\infty} \mathbb{L}^{-(\alpha\circ(f_2)_\infty)-\operatorname{ord}(K_{Y/X_2})} \\ &= \int_{Y_\infty} \mathbb{L}^{-(\alpha\circ(f_2)_\infty)-\operatorname{ord}(f_1^* K_{X_1/X_2})-\operatorname{ord}(K_{Y/X_1})} \\ &= \int_{(X_1)_\infty} \mathbb{L}^{-(\alpha\circ g_\infty)-\operatorname{ord}(K_{X_1/X_2})},\end{aligned}$$

giving us the birational transformation rule for g.

Finally, let $Y \to X$ be any proper birational morphism of smooth varieties. By the weak factorization theorem we can take a chain of blowups along smooth centers

$$\begin{tikzcd} & X_1 \arrow[dl] \arrow[dr] & & X_{m-1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & \\ Y = X_0 & & X_2 & & X_{m-2} & & X_m = X \end{tikzcd}$$

Note first that the rational map $X_{m-2} \dashrightarrow X$ is in fact a morphism: by the theorem, either there's some i such that for $j \geq i$ all the maps $X_j \dashrightarrow X$ are morphisms, or there's no such i (i.e., $i = m$), and then all the rational maps $X_j \dashrightarrow Y$ are morphisms, including for X_{m-2}; in this case, since $Y \to X$ is a morphism, we factor $X_{m-2} \to Y \to X$. In particular, we have by induction on length of the chain of morphisms (this uses the above step as the base case) that the birational transformation rule holds for $X_{m-2} \to X$ and $Y \to X_{m-2}$, and thus the result follows.

4 Applications

Kontsevich's theorem

We can now give an immediate proof of Kontsevich's theorem. We state it in the following form, which requires an auxiliary definition:

Definition. Two smooth varieties X_1, X_2 are K-equivalent if there exists a smooth variety Y and proper birational morphisms $Y \to X_1$, $Y \to X_2$, with $K_{Y/X_1} = K_{Y/X_2}$.

Theorem. *Two K-equivalent complex varieties X_1, X_2 have the same class in $\widehat{\mathcal{M}}_{\mathbb{C}}$, and thus the same Hodge numbers.*

Proof. Say Y is the smooth variety in the definition of K-equivalence, so

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 \end{tikzcd}$$

We know that

$$e(X_1) = \mu((X_1)_\infty) = \int_{(X_1)_\infty} \mathbb{L}^0$$

(in $\widehat{\mathcal{M}}_{\mathbb{C}}$, not in $K_0(\mathrm{Var}_{\mathbb{C}})$!). Applying the birational transformation rule, we can calculate this integral on Y as

$$\int_{Y_\infty} \mathbb{L}^{-\operatorname{ord}(K_{Y/X_1}}.$$

But by our assumption that $K_{Y/X_1} = K_{Y/X_2}$, this is the same as

$$e(X_2),$$

concluding the proof. □

Corollary. *Birational Calabi–Yau varieties have the same Hodge numbers.*

Proof. The only thing to show is that birational Calabi–Yau varieties are in fact K-equivalent. Say X_1, X_2 are birational Calabi–Yau varieties, and choose a common birational model of both, i.e., Y smooth projective with proper birational morphisms

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

Let us first note what the issue is: to show that X_1 and X_2 are K-equivalent, we need the relative canonical divisors K_{Y/X_1} and K_{Y/X_2} to be equal (as divisors!). We know that $K_{Y/X_1} = K_Y - f_1^* K_{X_1}$, and likewise for K_{Y/X_2}; by assumption that each $\omega_{X_i} = \mathcal{O}_X(K_{X_i})$, K_{X_1} is linearly equivalent to 0, as is K_{X_2}. Thus, we have that $K_{Y/X_1} \sim K_Y \sim K_{Y/X_2}$, i.e., the K_{Y/X_i} are linearly equivalent. However, we need actual equality as divisors! So, let $D = -K_{Y/X_1} + K_{Y/X_2}$. It's immediate that D is linearly equivalent to 0; in particular, D restricts to a degree-0 divisor on every curve $C \subset X$ (i.e., D is numerically trivial) and in particular *nef*.

This follows from a standard lemma in birational geometry, the negativity lemma:

Lemma. *Let $f : Y \to X$ be a proper birational morphism of smooth varieties. Let E be a divisor on Y with $-E$ f-nef (i.e., $\deg(-E|_C) \geq 0$ for every curve $C \subset Y$ contracted by f). Then E is effective if and only if $f_* E$ is.*

For a proof, see [KM98, Lemma 3.39]: the idea is to use induction on dimension to reduce to the case of the Hodge index theorem for surfaces.

We apply this in our setting as follows: since $D|_C$ has degree 0 for every curve C on Y, it's certainly true that $-D|_C$ has degree 0 for curves contracted by f_1, so $-D$ is f_1-nef. It's a standard fact that $(f_1)_*(K_{Y/X_1}) = 0$; thus we have that

$$(f_1)_*(D) = (f_1)_*(K_{Y/X_2}).$$

But K_{Y/X_2} is effective (it's defined locally be the determinant of the Jacobian matrix), so $(f_1)_* K_{Y/X_2}$ is as well, so the negativity lemma implies that D itself is effective, i.e., $K_{Y/X_1} \leq K_{Y/X_2}$. By the same argument applied to $-D$, we then get actual equality of divisors $K_{Y/X_1} = K_{Y/X_2}$, showing that X_1 and X_2 are K-equivalent. □

Motivic zeta functions

We'll finish by defining the motivic (Igusa) zeta function; there is another a motivic zeta function defined by Kapranov, so one should be cautious when reviewing the literature. We'll omit proofs and focus on sketching some results and conjectures.

For this section, X will be a smooth complex variety, and we'll work with the Grothendieck ring $\mathcal{M}_X$ instead of $\mathcal{M}_{\mathbb{C}}$ (i.e., we'll work with schemes over X, and remember their "structure" morphism to X). This will allow us to retain additional information, but can be ignored on first read.

Let X be a smooth variety and $D \subset X$ an effective divisor. We define Locally, one should just think of D as corresponding to the zero locus of a regular function $X \to \mathbb{A}^1$.

Definition. The motivic zeta function of D is the power series

$$Z_D(T) = \sum_m \mu(\mathrm{Cont}^m(D))T^m \in \mathcal{M}_X[[T]].$$

Note here that since $\mathrm{Cont}^m(D)$, and all its truncations to the finite-level jet schemes, are X-schemes (not just $\mathbb{C}$-schemes), we have that $e(\psi_\ell(\mathrm{Cont}^m(D)))\mathbb{L}_X^{-n\ell} \in \mathcal{M}_X$, and one can check that this is independent of $\ell \gg 0$, as before.

In fact, since X is a $\mathbb{C}$-scheme, there's a natural morphism $\alpha_* : \mathcal{M}_X \to \mathcal{M}_{\mathbb{C}}$ (not a ring map!), which sends $e(Y/X) \to e(Y/\mathbb{C})$ for any X-variety Y (i.e., just forget its X-variety structure). One can then check that $\alpha_*(\mu(\mathrm{Cont}^m(D)/X)) = \mu(\mathrm{Cont}^m(D)/\mathbb{C})$, so we can recover the usual motivic measure. We write $Z_{D,X}(T) = \alpha_*(Z_D(T)) \in \mathcal{M}_{\mathbb{C}}[[T]]$.

Remark. One can also think of $Z_D(T)$ as being

$$\int_{X_\infty} T^{\mathrm{Cont}(D)}.$$

Definition. For $i : W \subset X$ any morphism, we have a natural morphism $i^* : \mathcal{M}_X \to \mathcal{M}_W$, $e(Z) \mapsto e(Z \times_X W)$, which is in fact a ring map. In particular, when W is a subscheme of X, we have such a morphism, where an X-scheme $f : Z \to X$ is just mapped to $f^{-1}(W)$. For such a subscheme, we define

$$Z_{D,W}(T) := i^* Z_D(T) = \sum_m \mu\big(\mathrm{Cont}^m(D) \cap \psi_0^{-1}(W)\big)T^m \in \mathcal{M}_W[[T]].$$

Remark. In fact, the true motivic zeta function of [DL99] takes values in a Grothendieck ring of varieties with actions of the profinite group of roots of unity, in order to encode monodromy actions, but we will ignore this subtlety here.

The first thing we note is that the motivic zeta function is actually a rational function in T:

Theorem. *Say that $f : Y \to X$ is a log resolution of (X, D) (i.e., Y is smooth, f is proper birational, an isomorphism over $X - D$, and $\mathrm{Supp}(f^*(D)) \cup \mathrm{Exc}(f)$ is an snc divisor). Write $K_{Y/X} = \sum_{i\in I} k_i E_i$ and $f^*D = \sum_{i\in I} a_i E_i$. For $J \subset I$, write*

$$E_I^\circ = \Big(\bigcap_{j\in J} E_j\Big) - \Big(\bigcup_{j\notin J} E_j\Big)$$

(so the E_I° give a locally closed partition of Y). Then

$$Z_D(T) = \sum_{J\subset I} e(E_J^\circ) \prod_{j\in J} \frac{(\mathbb{L}-1)T^{a_j}}{\mathbb{L}^{k_j+1} - T^{a_j}}.$$

It's then clear that for $W \subset X$ a subscheme we have

$$Z_{D,W}(T) = \sum_{J\subset I} e(E_J^\circ \cap f^{-1}(W)) \prod_{j\in J} \frac{(\mathbb{L}-1)T^{a_j}}{\mathbb{L}^{k_j+1} - T^{a_j}}.$$

Example. Let X be a smooth surface and C a curve with a single node. Then a log resolution $f : Y \to X$ of C is obtained by blowing up X at the node; in this case, we have one exceptional divisor, E_1, with $k_1 = 1$ and $a_1 = 2$, and $E_2 = \widetilde{C}$, with $k_2 = 0$, $a_2 = 1$. We have that

$$\begin{aligned}
E_\emptyset^\circ &= Y - (E_1 \cup E_2) \cong X - C \\
E_{\{1\}}^\circ &= E_1 - \{\text{two points}\} \cong \mathbb{A}^1 - \{\text{pt}\} \\
E_{\{2\}}^\circ &= E_2 - \{\text{two points}\} \cong C - \{\text{point}\} \\
E_{\{1\cap 2\}}^\circ &= \{\text{two points}\}
\end{aligned}$$

The above expression then says that

$$\begin{aligned} Z_C(T) &= e(X-C) + (\mathbb{L}-1)^2 \frac{T^2}{\mathbb{L}^2 - T^2} + (e(C)-1)(\mathbb{L}-1)\frac{T^1}{\mathbb{L}-T} + 2(\mathbb{L}-1)^2 \frac{T^2}{\mathbb{L}^2-T^2}\frac{T^1}{\mathbb{L}-T} \\ &= e(X-C) + (e(C)-1)\frac{(\mathbb{L}-1)T}{\mathbb{L}-T} + \frac{(\mathbb{L}-1)^2T^2}{(\mathbb{L}-T)^2}. \end{aligned}$$

Definition. The topological zeta function of D is the rational function obtained by first evaluating $Z_D(T)$ (or rather its image in $\mathcal{M}_\mathbb{C}$) at $T = \mathbb{L}^{-s}$ (one can check this is well-defined, as it's the integral $Z_D(s) := \int_{X_\infty} \mathbb{L}^{-s\,\mathrm{ord}(D)}$.) One can then apply the Hodge–Deligne polynomial to get an element of $\mathbb{Z}[u, v, (uv)^{-1}]$, and finally evaluate at $u = v = 1$; this then gives a rational function in s, denoted $Z_D^{\mathrm{top}}(s)$. We also have the corresponding local version, $Z_{D,x}^{\mathrm{top}}(s)$.

Example. The topological zeta function of $D = V(x^2 + y^3) \subset \mathbb{A}^2$ is

$$\frac{4s+5}{(s+1)(6s+5)}.$$

Remark. In terms of the above data of a log resolution, this gives the rational function

$$\sum_{J \subset I} \chi^{\mathrm{top}}(E_J^\circ) \prod_{j \in J} \frac{1}{sa_j + k_j + 1},$$

where χ^{top} is the (compactly supported) topological Euler characteristic. Note that if one attempted to simply define the topological zeta function as the above expression, it wouldn't be clear it's independent of the choice of log resolution, and motivic integration provides a natural way to see this.

Conjecture (Vey)**.** *If s_0 is a pole of $Z_{D,x}^{\mathrm{top}}(s)$ of order $\dim X$, then $s_0 = -\mathrm{lct}_x$.*

Theorem (Nicaise)**.** *This is true even for poles of $Z_{D,x}(s)$ of order $\dim X$, and thus Vey's conjecture is true.*

Definition. Given a polynomial $f \in \mathbb{A}^n$, there is a monic polynomial $b_f(s) \in \mathbb{Q}[s]$, such that $b_f(s)$ is minimal among the polynomials $b(s)$ such there's some differential operator D with $D \cdot f^{s+1} = b(s) f^s$. This extends to a divisor D in a smooth complex variety X.

Conjecture (monodromy conjecture)**.** *In the above setting, $Z_{D,X}(T)$ lies in the subring of $\mathcal{M}_k[[T]]$ generated by $\mathcal{M}_k$ and elements of the form*

$$\frac{(\mathbb{L}-1)T^N}{\mathbb{L}^v - T^N},$$

where v and N are positive integers such that:

1. *(weak version) $\exp(-2\pi i v/N)$ is an eigenvalue for the monodromy action of the Milnor fiber at some point $x \in D$.*
2. *(strong version) $-v/N$ is a root of the Bernstein–Sato polynomial of D.*

This would imply a corresponding version for the topological zeta function.

Example. The Bernstein–Sato polynomial for $x^2 + y^3$ is

$$(s+1)(s+5/6)(s+7/6).$$

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