Day 1

Plan Day 1: Intro, chow gp, def to normal cone

Day 2: int prod + chow ring

Day 3: examples

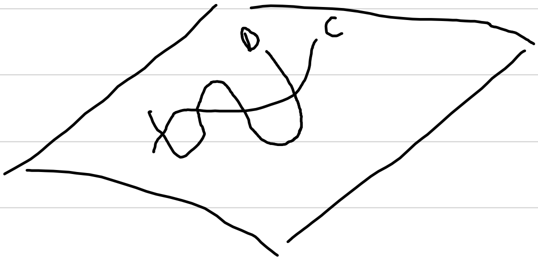
Day 4: chern classes, deg loci

Day 5: examples w/ Grassmannian

note: $V, W \subset Y$ schemes, $V \cap W$ = scheme-theoretic intersection

Recall Let Y = sm proper surface

$C, D \subset Y$ sm curves



* if $C \pitchfork D$, then we say $(C.D) = \#|C \cap D|$

* if $C \not\pitchfork D$ then we say $(C.D) = \sum_{x \in C \cap D} \text{mult}_x C \cap D$

For general curves $C, D \subset Y$ (i.e singular), we can still define the intersection # to extend the prev case:

$$(C.D) := \deg(\mathcal{O}_C(D)) = \deg(\mathcal{O}_D(C))$$

which is independent of linear equiv. class.

E.g (Bézout) $C, D \subset \mathbb{P}^2$ deg c, d resp.

Then $(C.D) = cd$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ f^{-1}D \arrow[u, phantom, "\cup"] \arrow[r] & D \arrow[u, phantom, "\cup"] \end{tikzcd}$$

(proper) map of sm proper surfaces

curve

Assume $C \not\subseteq f^{-1}D$

C = irred

If $C < X$ a curve, can ask $(f^{-1}D . C) = ?$

Case 1 $f(C) = pt \Rightarrow C \cap f^{-1}D = \emptyset$

$\Rightarrow (f^{-1}D . C) = 0$

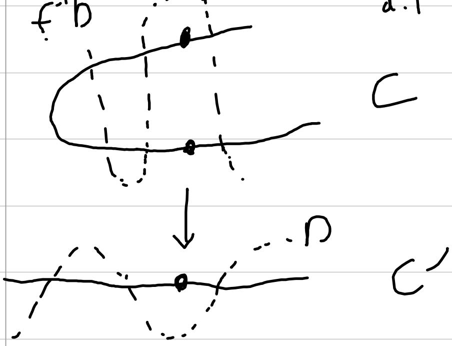
Case 2 $f : C \twoheadrightarrow C' = f(C)$ is a curve

C', C = irred $\Rightarrow f$ = finite gen. flat

$\Rightarrow d = \deg(f)$

$\forall y \in C'$ generic $= \sum_{x \in f^{-1}(y)} \text{mult}_x f^{-1}(y)$

Assume D intersects C' at generic pts OR replace $C \to C'$ with normalization $\tilde{C} \xrightarrow[d:1]{} \tilde{C}'$



$$\begin{tikzcd} f^{-1}D \cap C \arrow[d, "d:1"'] \arrow[r, phantom, "\subset"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "d:1"] \\ D \cap C' \arrow[r, phantom, "\subset"] & C' \end{tikzcd}$$

$$(f^{-1}D . C) = \sum_{x \in f^{-1}D \cap C} \text{mult}_x f^{-1}D \cap C$$

$$= d \sum_{y \in D \cap C'} \text{mult}_y D \cap C'$$

$$= d (D . C') =: \deg(C/f(C)) (D . f(C))$$

Recall If we think of $[C] \in Cl(X)$, then $f_*[C] := \deg(C/f(C))[f(C)]$

So we are getting $(f^{-1}D . [C]) = (D . f_*[C])$

We can this the proj formula since it is (not coincidentally) like proj formula for v. bdles.

So, what's the point?. Take field $k = \bar{k}$.

We denote Scheme := f.g sch / k

Variety := red. irred. f.g sch / k

We want to generalize the intersection theory on sm proper surfaces to arbitrary schemes and certain/arbitrary intersections of subvarieties.

Want: For Y = sm var, a ab gp $A^*(Y)$ that encodes information with intersection product s.t

(1) int prod is commutative
i.e intersecting C w/ D is same as intersecting D w/ C

(2) $[Y] = \underline{1}$

(3) Functorial pullback resp. int. prod.

(4) Functorial proper pushforwards

(5) proj formula

(6) if V & W intersect "nicely" enough (e.g transversally, if smooth)
$[V] \cdot [W] = [V \cap W]$

Convention

$$\begin{tikzcd} E = \operatorname{Spec}(\operatorname{Sym}^{\bullet}_{\mathcal{O}_X} \mathcal{E}^{\vee}) \arrow[d] \\ X \end{tikzcd}$$

if $F = \operatorname{Spec}(\operatorname{Sym}^{\bullet}_{\mathcal{O}_X} \mathcal{F}^{\vee})$ v. bdle, then $\mathcal{E}^{\vee} \to \mathcal{F}^{\vee}$ corresp $F \to E$

$$\begin{tikzcd} \mathbb{P}E = \operatorname{Proj}(\operatorname{Sym}^{\bullet}_{\mathcal{O}_X} \mathcal{E}^{\vee}) = \text{lines in } \overset{\mathcal{E}}{E} \arrow[d, "\pi"'] \\ X \end{tikzcd}$$

$\pi^* \mathcal{E}^{\vee} \twoheadrightarrow \mathcal{O}(1)$

corresp. $\mathcal{O}(-1) \rightarrowtail E$

Goal 1: Generalize $cl(Y)$ to $A_*(Y)$

Want $A_*(Y)$ to encapsulate data for every subscheme $V \subset Y$

$$Z_*(Y) = \bigoplus_{k \geq 0} Z_k(Y)$$

Need $A_*(\ )$ to be functorial on

- flat morph: graded of deg m flat pullback

$f: X \to Y$
flat rel dim m

$$f^*: A_*(Y) \to A_{*+m}(X)$$
$$[V] \mapsto [f^{-1}V]$$

- proper morph: graded of deg 0 proper pushforward sat. proj formula

$f: X \to Y$
proper

$f|_V : V \twoheadrightarrow f(V) = W$

if $\dim(W) < \dim(V)$, then $f_*[V] = 0$.

otherwise $f|_V$ is gen fin.

$$\Rightarrow \quad f_*[V] \overset{\text{func.}}{=} (f|_V)_*[V] = (f|_V)_*[f|_V^{-1}W] \overset{\text{p.f.}}{=} \deg(V/W)\,[W]$$

Need $A_*(Y) = Z_*(Y)/\sim$ smallest equiv relation so that above will be true,

$$\begin{tikzcd}
V_t \subset \underset{\dim k+1}{V} \arrow[r, hook, "i", "\text{closed}"'] \arrow[rr, bend left, "f"] \arrow[dr, bend right] & Y \times \mathbb{P}^1 \arrow[r, two heads, "pr_2"'] \arrow[d, "pr_1"] & \mathbb{P}^1 \ni t \\
 & Y &
\end{tikzcd}$$

if f is surj, then is flat so have

$$\begin{tikzcd}
{[0] = [\infty]} \in A_0(\mathbb{P}^1) \arrow[r, "f^*"] \arrow[drr, mapsto, bend right] & A_k(V) \arrow[r, "i_*"] & A_k(X \times \mathbb{P}^1) \arrow[d, "pr_{1*}"] \\
 & & {[v_0] = [v_\infty]} \in A_k(Y)
\end{tikzcd}$$

So whenever $\alpha, \beta \in A_k(Y)$ arise as $\alpha = V_0, \beta = V_\infty$ for such a $\mathcal{V}$ as above

$\alpha \sim \beta$ rat equiv.

rmk There is alt but equiv. def of rat equiv that directly gen. the lin. equiv of Weil Divisors you learned in AG [see F 1.3]

Ex $A_{n-1} = Cl$

Ex $A_*(Y) = A_*(Y_{red})$

$$[Y] = \sum_{Y_i} \ell(\mathcal{O}_{Y,Y_i})[Y_i]$$

A_* behave just as cl !

Ex 1 (Excision) $V \underset{\substack{i \\ \text{subsch}}}{\subset} Y \underset{\substack{j \\ \text{open}}}{\supset} U = Y \setminus V$

$$A_*(V) \xrightarrow{i_*} A_*(Y) \xrightarrow{j^*} A_*(U) \to 0$$

Ex 2 $A_*(\mathbb{A}^n) = 0$

more gen.
$$\begin{tikzcd} E \arrow[d, "\pi"] \\ Y \arrow[u, bend left, "s_E"] \end{tikzcd}$$
E rk e $\quad \pi^*: A_*(Y) \xrightarrow{\sim} A_{*+e+1}(E)$

Call $s_E^* := (\pi^*)^{-1}$ the <u>gysin hom</u>

Ex 3 $i: D \subset Y$ cartier

$$i^*: A_*(Y) \longrightarrow A_{*-1}(D)$$
$$[V] \longmapsto D|_V$$
$:=$ cartier divisor s.t. $\mathcal{O}_V(D|_V) = \mathcal{O}_V(D)$

call i^* the <u>gysin hom</u>

$V \not\subset |D| \implies i^*[V] = [V \cap D]$

$V \subset |D|$, $\underline{\mathcal{O}_D(D) = \mathcal{O}_D} \implies i^*[V] = 0$

Deformation to normal cone

$X \subset Y$ closed emb. of schemes

Normal cone $C_X Y := \mathrm{Spec}\left(R(I_X, \mathcal{O}_X) / I_V \cdot R(I_X, \mathcal{O}_X) \right)$

$$\begin{tikzcd}
\mathbb{P}C_X Y = E \arrow[r, phantom, "\overset{\text{cartier}}{\subset}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Bl_X Y = \mathrm{Proj}(R(I_X, \mathcal{O}_X)) \arrow[d, "p"] \\
X \arrow[r, phantom, "\subset"] & Y
\end{tikzcd}$$

Want:

$$\begin{tikzcd}
X \times \mathbb{P}^1 \arrow[r, phantom, "\subset"] \arrow[dr, "pr"'] & M := M^\circ_X Y \arrow[d, "q"] \\
& \mathbb{P}^1
\end{tikzcd}$$

$\leftarrow$ flat

over $\mathbb{P}^1 - \{\infty\}$

$$\begin{tikzcd}
X \times \mathbb{A}^1 \arrow[r, phantom, "\subset"] & Y \times \mathbb{A}^1 = q^{-1}(\mathbb{P}^1 - \{\infty\}) \arrow[d, "q = pr"] \\
& \mathbb{P}^1 - \{\infty\}
\end{tikzcd}$$

over $\infty \in \mathbb{P}^1$

$$\begin{tikzcd}
X \arrow[r, hook, "s", "\text{zero section}"'] & C_X Y = M_\infty \arrow[d] \\
& \{\infty\}
\end{tikzcd}$$

Why do we want this?

- $[Y] = [C_X Y]$ in $A_*(M)$

  enumerative properties, e.g. GRR
- Properties of $X \subset Y$ reduce to case $X \overset{s}{\hookrightarrow} C_X Y$

$\rightsquigarrow$ can construct gen of int theory

Ex 4 using prev. Ex 1-3 :

$$\begin{tikzcd}
 & & A_*(M_\infty) \arrow[d, "\varphi_{\infty *}"] & \\
{[V \times \mathbb{P}^1]}, & {[M^0_{X \cap V} V]} \in & A_*(M = M^0_X Y) \arrow[r, "\varphi_\infty^*"] \arrow[d] & A_*(M_\infty) \\
 & A_*(Y) = & A_*(M \setminus M_\infty = Y \times \mathbb{A}^1) \arrow[d] \arrow[ur, dashed, "\sigma"'] & \\
 & & 0 &
\end{tikzcd}$$

$V \subseteq X$ $V \nsubseteq X$

$$\varphi_\infty^* \varphi_{\infty *}[M_\infty] = \varphi_\infty^*[M_0]$$
$$= [M_\infty \cap M_0]$$
$$= 0$$
$$\Rightarrow \mathcal{O}_{M_\infty}(M_\infty) = \mathcal{O}_{M_\infty}$$
$$\Rightarrow \forall\, V \subset M_\infty \quad \varphi_\infty^*[V] = 0$$

call $\sigma : A_*(Y) \longrightarrow A_*(M_\infty) = A_*(C_X Y)$

$$[V] \longmapsto \begin{cases} [V] & V \subseteq X \\ [C_{X \cap V} V] & \text{else} \end{cases}$$

from $V \subset X \overset{s}{\hookrightarrow} C_X$ zero section

Specialization map

Construction

$$\begin{tikzcd}
\mathbb{P}C \arrow[r, phantom, "\overset{z=0}{\underset{\text{hyperplane}}{\subset}}"] & \mathbb{P}(C \oplus 1) = \mathrm{Proj}(R^\bullet[z]) =: \overline{C} & \\
 & & \\
\mathbb{P}(1) = X \arrow[uur, "\text{closed}"] \arrow[rr, "s"'] & & C = \mathbb{P}C \oplus 1 = \mathbb{P}(C \oplus 1) \setminus \mathbb{P}C \arrow[uul, phantom, "\cup \ \text{open}"]
\end{tikzcd}$$

$C = \mathrm{Spec}(R^\bullet)$

$R^\bullet$ f.g by R^1

deg 1 for $\mathcal{O}_X$-coord

$\mathbb{P}C \cap \mathbb{P}1 = \emptyset$

ex $C = C_X Y$, E

$$\begin{tikzcd}
C_X Y \arrow[r, phantom, "\underset{open}{\subset}"] \arrow[dd, bend right, "\pi"'] & E=\mathbb{P}(C_{X\times\{\infty\}}Y\times\mathbb{P}^1)=\mathbb{P}(C_XY\oplus\mathbb{1}) \arrow[d, phantom, "\cup"] \arrow[dr, bend left] & & \\
X\times\{\infty\} \arrow[u, "s"] \arrow[r, phantom, "\subset"] \arrow[d, equal] & Bl_{X\times\{\infty\}}X\times\mathbb{P}^1 \arrow[r, phantom, "\subset"] \arrow[d, equal] & \bar{M}:=Bl_{X\times\{\infty\}}Y\times\mathbb{P}^1 \arrow[d, "\rho"] \arrow[ddr, bend left, "q"] & \\
X\times\{\infty\} \arrow[r, phantom, "\overset{Cartier}{\subset}"] \arrow[dr, equal] & X\times\mathbb{P}^1 \arrow[r, phantom, "\subset"] & Y\times\mathbb{P}^1 \arrow[d, "pr_1"'] \arrow[dr, "pr_2"] & \\
& X \arrow[r, phantom, "\subset"] & Y & \mathbb{P}^1
\end{tikzcd}$$

$$pr_1^* I_X \oplus pr_2^* m_\infty \twoheadrightarrow I_{X\times\{\infty\}}$$

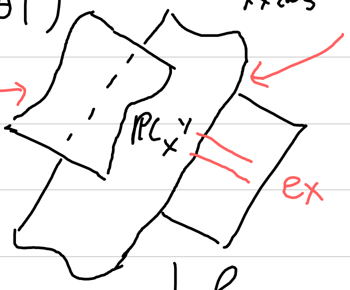
$$\Longrightarrow \quad C_{X\times\{\infty\}}Y\times\mathbb{P}^1 = C_X Y\oplus\mathbb{1}$$

Over $\infty\in\mathbb{P}^1$:

$$\bar{M}_\infty = \rho^{-1}(Y\times\{\infty\})$$

$$\|$$

$$E=\mathbb{P}(C_XY\oplus 1) + \tilde{Y} = Bl_{X\times\{\infty\}}Y\times\{\infty\} = Bl_X Y$$



ex of $Bl_X Y = E\cap\tilde{Y}$

$$\begin{tikzcd}
E \arrow[d, "\pi"] & \bar{M}_\infty \arrow[d, "\rho_\infty"] \\
X\times\{\infty\} \arrow[r, phantom, "\subset"] & Y\times\{\infty\}
\end{tikzcd}$$

$Y\times\{\infty\} \underset{Cartier}{\subset} Y\times\mathbb{P}^1$

$\bar{M}_\infty = \tilde{Y} + aE$.

Cartier in $\bar{M}$

$((-E)^{(n-1)}\cdot\tilde{Y}) = -mult_X Y$

$\|$ using proj form. $\quad \|$

$a((-E)^n) = -mult_{X\times\{\infty\}} Y\times\mathbb{P}^1$

$\Longrightarrow a=1$

$n=codim_Y(X)$

Let $M:=M_X^o Y = \bar{M}\setminus\tilde{Y}$

Day 2

Recap (1) Talked about what properties we want our int theory

(2) defined $A_*(Y) = Z_*(Y)/\sim_{rat}$

- flat pullback $f^*[V] = [f^{-1}V]$
- proper pushward $f_*[V] = \begin{cases} 0 & \dim(f(V)) < \dim(V) \\ \deg(V/f(V))[f(V)] & \text{else} \end{cases}$
- extend cl + its properties
- gysin hom for
  - $A_*(E) \xrightarrow{s_E^*} A_*(Y)$
  - $A_*(Y) \xrightarrow{\iota^*} A_*(D)$

(3) Def to normal cone

Specialization map $\sigma: A_*(Y) \longrightarrow A_*(C_X Y)$

$$[V] \longmapsto \begin{cases} [V] & |V| \le |X| \\ [C_{X\cap V} V] & |V| \not\le |X| \end{cases}$$

Note: $A_*(Y)$ is graded by dim ; f^* deg = rel. dim

f_* deg 0

$A^*(Y) = A_{n-*}$ graded by codim

$$D, W \longmapsto \deg(D \cap W)$$

Goal 2 : Generalize $\mathrm{Cart}(Y) \times Cl(X) \longrightarrow \mathbb{Z}$

Think of as $\mathrm{Cart}(Y) \otimes_{\mathbb{Z}} A^1(X) \xrightarrow{\cdot} A_0(X) \xrightarrow{\deg} \mathbb{Z}$

$$\sum n_i [y_i] \longmapsto \sum n_i$$

$$\mathrm{Cart}(Y) \subseteq Z_1(Y)$$

Cartier div $\rightsquigarrow$ LCI of codim d

(LCI of codim 1)

Thm 2.1 / Def (refined Gysin hom's) $i: V \subset Y$ LCI subsch of codim d, then there is a unique collection $i^!$ of gp hom

$$i_f^{!\,(k)} : A_k(X) \longrightarrow A_{k-d}(X \times_Y V)$$

for any $X \xrightarrow{f} Y$ var/Y and k s.t

Proper pushforward (a)

$$\begin{tikzcd} X' \arrow[rr, "p"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

proper morph of var/Y

proj formula

then

$$\begin{tikzcd} X' \times_Y V \arrow[r, phantom, "\subset"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \boxed{X'} \arrow[d, "p"] \\ X \times_Y V \arrow[r, phantom, "\subset"] & X \end{tikzcd}$$

$$i_f^! \, p_* = p'_* \, i_g^!$$

flat pullback (b)

$$\begin{tikzcd} X' \arrow[rr, "p"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

flat morph of rel dim n of var/Y

a gysin hom for $i: V \to Y$ flat (e.g prop 1.7)

then

$$\begin{tikzcd} X' \times_Y V \arrow[r, phantom, "\subset"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "p"] \\ X \times_Y V \arrow[r, phantom, "\subset"] & \boxed{X} \end{tikzcd}$$

$$i_g^! \, p^* = p'^* \, i_f^!$$

Commutativity (c)

$$\begin{tikzcd} V' \times_X W' \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\lrcorner", very near start] & W' \arrow[r] \arrow[d, phantom, "\cap"] \arrow[d, "j'"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, phantom, "\cap"] \arrow[d, "j"] \\ V' \arrow[r, phantom, "\subset"] \arrow[r, "i'"'] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \boxed{X} \arrow[r, "g"] \arrow[d, "f"] & T \\ V \arrow[r, phantom, "\subset"] \arrow[r, "i"'] & Y & \end{tikzcd}$$

$$j^!_{gi'} \, i^!_f = i^!_{fj'} \, j^!_g$$

Ex comm. of int prod $X = T = Y$

Compatibility (d)

$$\begin{tikzcd} V' \times_X X \arrow[r, hook, "i'"] \arrow[d] & X \arrow[d, "F"] \\ V \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

if i' is LCI of codim d then $i^! = i'^!$ for all Var/X

e.g transverse intersections

normalization (e) $i^!_{id_Y}[Y] = [V]$

Note: Only need (a), (b), (d), (e) for uniqueness.

Note When we have $V \subset \underset{}{\overset{\underset{\| f = id_Y}{Y}}{Y}}$ then we denote $i^* := i^!_{id_Y}$ and call Gysin hom.

Rmk 2.2 Ref. gysin hom extends gysin hom for case of divisors and for zero section of vector bundle.

compatibility

pf.

$$\begin{tikzcd}
V \arrow[r, hook, "s_E"] \arrow[d, phantom, "\cap"] & E|_V \arrow[r, "\pi"] \arrow[d, phantom, "\cap"] & V = \text{subvar} \arrow[d, phantom, "\cap"] \\
X \arrow[r, hook, "s_E"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d, equal] & X \arrow[d, equal] \\
X \arrow[r, hook, "s"'] & E \arrow[r, "\pi"'] & X
\end{tikzcd}$$

$$\xrightarrow{\text{compatibility}} \quad s_E^! \overbrace{\pi^*[V]}^{[E|_V]} = s_E^*[E|_V]$$

$$= [V]$$

$$\overset{\text{norm.}}{=} (\pi^*)^{-1}\pi^*[V] \qquad \square$$

Cor 2.3 (Functoriality) $V \underset{j}{\subset} W \underset{i}{\subset} Y$

associativity

LCI codim e LCI codim d

then $j^! i^! = (ij)^!$

pf Show $j^! i^!$ sat. (a), (b), (d), (e). For (d):

$$\begin{tikzcd}
V' \arrow[r, hook, "j'"] \arrow[d] & W' \arrow[r, hook, "i'"] \arrow[d] & X \arrow[d, "f"] \\
U \arrow[r, hook, "j"'] & W \arrow[r, hook, "i"'] & Y
\end{tikzcd}$$

Let $x \in V'$. Suppose

$(x_1, \dots, x_d)$ = reg seq gen $\mathcal{I}_{W/Y} \cdot \mathcal{O}_{Y,f(x)}$

$(x_1, \dots, x_d, x_{d+1}, \dots, x_{d+e})$ = reg seq gen $\mathcal{I}_{V/Y} \cdot \mathcal{O}_{Y,f(x)}$

$I \subset R$ = local noeth

" gen by reg seq $(x_1, \dots, x_d) = \underline{x}$

If $(y_1, \dots, y_d) = \underline{y}$ another seq of minimal generators,

$$\begin{tikzcd}
R^d \arrow[r, "\underline{x}"] \arrow[d, "\wr"] & R \\
R^d \arrow[ur, "\underline{y}"'] &
\end{tikzcd}$$

$K^\bullet(\underline{x}) \simeq K^\bullet(\underline{y})$

$\Rightarrow H^\bullet(\underline{y}) = R/I$

$\Rightarrow \underline{y}$ = reg seq

if $\mathcal{I}_{V/Y} \cdot \mathcal{O}_{X,x}$ gen by reg seq of length $d+e$, then $(x_1, \dots, x_d, x_{d+1}, \dots, x_{e+d})$ reg seq in $\mathcal{O}_{X,x}$

BUT any subset of this seq is a reg seq, in particular $(x_1, \dots, x_d)$ reg seq in $\mathcal{O}_{X,x}$.

So $i'j'$ = LCI of codim $d+e$

implies i' = LCI codim e

j' = LCI. codim d

Q Assuming ref. Gysin hom exist, what must they be ?

$$\begin{tikzcd} W := X \times_Y V \arrow[r, hook, "i'"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] & \dim n \\ V \arrow[r, hook, "i"'] & Y & \text{LCI codim } d \end{tikzcd}$$

We'll use (a) $\Longrightarrow$ reduce to case $\alpha = [X] \in A_n(X)$

then (a)+(d)+(e) on def to normal cone:

$$\begin{tikzcd} W \times \{t\} \arrow[r, hook] \arrow[d, hook, "t"'] \arrow[ddd, bend right=60, "g"'] & M'_t \arrow[d, hook, "\psi'_t"] \arrow[ddd, bend left=60, "\tilde{f}_t"] & \\ W \times \mathbb{P}^1 \arrow[r, hook, "\chi'"] \arrow[d, "g \times id"'] \arrow[dr, phantom, "\lrcorner", very near start] & M^\circ_W X = M' \arrow[r, "q'"] \arrow[d, "\tilde{f}"] & \mathbb{P}^1 \\ V \times \mathbb{P}^1 \arrow[r, hook, "\chi"] & M^\circ_V Y = M \arrow[ur, "q"'] & \\ V \times \{t\} \arrow[r, hook] \arrow[u, hook, "t"] & M_t \arrow[u, hook] & \end{tikzcd}$$

Over $t \neq \infty$:

$$\begin{tikzcd} W \times \{t\} \arrow[r, "i'"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times \{t\} = M'_t \arrow[d, "f = \tilde{f}_t"] \arrow[dd, bend left=60, "\tilde{f}\psi'_t"] \\ V \times \{t\} \arrow[r, hook, "i"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times \{t\} = M_t \arrow[d, hook, "\psi_t"] \\ V \times \mathbb{P}^1 \arrow[r, hook, "\chi"'] & M \end{tikzcd}$$

$\overset{\text{compatibility}}{\Longrightarrow}$ over $t \neq \infty$: $\chi^!_{\tilde{f}\psi'_t}[M'_t] = i^![X]$

But proper pushforward $t_* i^![X] = t_* \chi^!_{\tilde{f}\psi'_t}[M'_t] \overset{pr}{=} \chi^!_{\tilde{f}}[M'_t]$

$$\| $$

$\infty_* \chi^!_{\tilde{f}\psi'_\infty}[M'_\infty] \overset{pr}{=} \chi^!_{\tilde{f}}[M'_\infty]$

Recall $[M'_t]$ in $A_*(M')$ is indep of t by flatness of q & $\sim_{rat}$ of pts of $\mathbb{P}^1$

So, what's $\chi^!_{\tilde{f}\psi'_\infty}[M'_\infty]$?

over $t=\infty$:

$$\begin{tikzcd}
W\times\{\infty\} \arrow[r, hook, "s"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & C_W X = M'_\infty \arrow[d, "\tilde{f}_\infty"] \\
V\times\{\infty\} \arrow[r, hook, "s_N = \chi_\infty"] \arrow[d, hook, "\infty"'] \arrow[dr, phantom, "\lrcorner"] & N_V Y = M_\infty \arrow[d, hook, "\psi_\infty"] \\
V\times\mathbb{P}^1 \arrow[r, hook, "\chi"'] & M
\end{tikzcd}$$

$$\xRightarrow{\text{compatibility}} \quad \chi^!_{\psi_\infty \tilde{f}_\infty}[C_W X] = (s_N)^!_{\tilde{f}_\infty}[C_W X]$$

$$\Big/\!\!\Big/ \ \text{proper}$$

$$\begin{tikzcd}
W \arrow[r, hook, "s"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & C_V X \arrow[d, phantom, "\cap"] \arrow[ddr, bend left, "\tilde{f}_\infty"] & \\
W \arrow[r, hook, "s_{g^*N}"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner"] & g^*N \arrow[d, "g"] & \\
V \arrow[r, hook, "s_N"'] & N_V Y = N &
\end{tikzcd}$$

$$\xRightarrow{\text{compatibility}} \quad (s_W)^!_g [C_W X] = s^*_{g^*N}[C_W X]$$

gysin hom for zero section of v. bdle

note:

$g^* I_V \twoheadrightarrow I_W$

$\Rightarrow g^* \bigoplus I_V^i / I_V^{i+1} \twoheadrightarrow \bigoplus I_W^i / I_W^{i+1}$

$\Rightarrow C_W X \underset{\text{closed}}{\subset} g^* N_V$

Hence, $\infty_* s^*_{g^*N}[C_W X] = t_* i^![X]$

$$\Big\Downarrow\ pr_{1*} \quad \text{functoriality}$$

$$\begin{tikzcd}
W\times\{t\} \arrow[d, hook, "t"] \arrow[dd, bend right=60, "id"'] \\
W\times\mathbb{P}^1 \arrow[d, "pr_1"] \\
W
\end{tikzcd}$$

$s^*_{g^*N}[C_W X] = i^![X]$

concl

$$\begin{tikzcd} i_F^{!(k)} : A_k(X) \arrow[r, dashed] \arrow[d, "\sigma"'] & A_{k-d}(X \times_Y V) \\ A_k(C_{X\times_Y V}X) \arrow[r] & A_k(f'^*N_V Y) \arrow[u, "s^*_{f'^*N}"'] \end{tikzcd}$$

for any $f: X \longrightarrow Y$ var/Y and k.

Ex $s : X \hookrightarrow E$ section of v.bdle E/X of rk e

$$\begin{tikzcd} Z(s) \arrow[r, hook] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, hook, "s"] \\ X \arrow[r, hook, "s_E"] & E \end{tikzcd}$$

$$\mathbb{Z}(s) := \left(s_E^{!}\right)_s^{(e)} [X] \in A_{n-e}(Z(s))$$

if s = reg section, mult results $\Rightarrow$ $\mathbb{Z}(s) = [Z(s)]$

Y = sm var, dim n

The nice intersection theory on Smooth Varieties

$f : X \to Y$
dim m

$$X \xhookrightarrow[\Delta_f]{} X \times Y$$

why is this nice?

- scheme-theoretically $\Delta^{-1}(V \times W) = V \cap f^{-1}W$
- LCI of codim n
- $N_{X\times Y} X = f^* T_Y$

Int. Prod.

$$\begin{tikzcd}
y, x \arrow[rr, mapsto] & & \boxed{f^*y \cap x := \Delta_f^*(x \times y)} \\
A^p(Y) \otimes_{\mathbb{Z}} A_k(X) \arrow[r] & A_{n+k-p}(X \times Y) \arrow[r, "\Delta_f^*"] & A_{k-p}(X) \\
A^p(|y|) \otimes_{\mathbb{Z}} A_k(|x|) \arrow[r] \arrow[u] & A_{n+k-p}(|x| \times |y|) \arrow[r, "\Delta_f^!"] \arrow[u] & A_{k-p}(|x| \cap f^{-1}|y|) \arrow[u] \\
y, x \arrow[rr, mapsto] & & \Delta_f^!(x \times y)
\end{tikzcd}$$

hence $\operatorname{supp}(f^*y \cap x) = |x| \cap f^{-1}|y|$

Pullback $f^* : A_*(Y) \to A_*(X)$

$$y \longmapsto f^*y \cap [X]$$

↳ functorial (because of cor 2.3)

Prop 2.6 If $V \subset Y$ subvar of codim d and $f^{-1}V \subset X$ is LCI of codim d as well, then $f^*[V] = [f^{-1}V]$.

pf

$$\begin{tikzcd}
f^{-1}V \arrow[r, hook] \arrow[d, "j"] & X \times V \arrow[d, "f \times j"] & \text{LCI codim } n \\
X \arrow[r, hook, "\delta_f"] & X \times Y
\end{tikzcd}$$

$$\Rightarrow f^*[V] = \Delta_f^![X \times V] \overset{\text{compatibility}}{=} [f^{-1}V] \qquad \square$$

Cor 2.7 If f is flat, then f^* is same as prev. defined pullback for flat since flat pullbacks preserve regularness of sequences.

If $X=Y$, then we get

$$A^p(Y)\otimes A^q(Y)\longrightarrow A^{p+q}(Y)$$
$$\alpha,\beta\longmapsto \alpha\cdot\beta := \Delta^*(\alpha\times\beta)$$

This makes $A^*(Y)$ into a graded unital comm. ring s.t.

(a) Proj formula; i.e $f_*(f^*y\cap x)=y\cap f_*x$

Result of more gen assoc & comm for arb. $f: X\to Y$
Fulton Prop 8.1.1a (or b)
(i) $X\to Y=Y$
(ii) $X=X\to Y$

(b) (i) $f^*(y_1\cdot y_2)\cap x = f^*y_1\cap(f^*y_2\cap x)$

(ii) when $X=$ sm, $f^*y\cap(x_1\cdot x_2)=(f^*y\cap x_1)\cdot x_2$

In particular when X is smooth

$$f^*(y_1\cdot y_2)\cap[X]=f^*y_1\cap(f^*y_2\cap[X])$$
$$=(f^*y_1\cap[X])\cdot(f^*y_2\cap[X])$$

$\rightsquigarrow f^*: A^*(Y)\to A^*(X)$ graded ring hom

(c) if V, W intersect generically transversally, then $[V]\cdot[W]=[V\cap W]$

pf associativity comes from functeriality of gysm on

$$\begin{tikzcd}
 & Y\times Y \arrow[dr, hook, "id\times\Delta"] & \\
Y \arrow[ur, hook, "\Delta"] \arrow[r, hook, "\Delta"'] & Y\times Y \arrow[r, hook, "\Delta\times id"'] & Y\times Y\times Y
\end{tikzcd}$$

Day 3 rmk to Day 2 :

$$\underset{\dim k}{V}, \underset{\dim r}{W} \subset Y \quad \dim n$$

when $V \cap W$ has dim $k+r-n$,

$$[V] \cdot [W] = \sum_{\substack{Z \subset V \cap W \\ \text{irred} \\ \text{comp}}} \underbrace{i(V,W;Z)}_{\text{intersection multiplicity}} \cdot [Z]$$

Thm For $Z \subseteq V \cap W$ irred component of dim $k+r-n$,

$$1 \leq i(V,W;Z) \leq \ell(\mathcal{O}_{V \cap W, Z})$$

with equality if $\mathcal{O}_{V \cap W, Z}$ is CM.

note: $[V \cap W] = \sum \ell(\mathcal{O}_{V \cap W, Z}) [Z]$

so, we don't always have $[V] \cdot [W] = [V \cap W]$

BUT when V intersects W generically transversally, for example, we do and in fact $[V] \cdot [W] = [V \cap W] = [\underset{\substack{\uparrow \\ \text{set-theoretic intersection}}}{|V \cap W|}]$

Ex $A_*(\mathbb{P}^n) = \mathbb{Z}[h]/h^{n+1}$ $\quad$ h = hyperplane class

use ind on n + Excision

$$A_*(X\times\mathbb{P}^n) = A_*(X)[\beta]/\beta^{n+1} \qquad \beta = pr_2^* h$$

char(k)=0 $\quad f: \mathbb{P}^r \longrightarrow \mathbb{P}^s$ $\qquad \Gamma_f \subset \mathbb{P}^r\times\mathbb{P}^s$

$$A_*(\mathbb{P}^r\times\mathbb{P}^s) = \mathbb{Z}[\alpha,\beta]\big/\alpha^{n+1},\beta^{n+1}$$

$$[\Gamma_f] = \sum_{i=0}^{r} c_i\,\alpha^{r-i}\beta^{s-r+i}$$

given by $V \subset H^0(\mathbb{P}^r,\mathcal{O}(d))$ dim $s+1$, basept free $d=\deg(f)$

$$c_i = \deg\left([\Gamma_f]\cdot\alpha^i\beta^{r-i}\right) \overset{\text{Kleiman-Bertini}}{=} \left|\Gamma_f \cap pr_1^{-1}\underset{\text{codim } i \text{ lin space}}{L_i} \times pr_2^{-1}L_{r-i}\right|$$

$$= L_i \cap f^{-1}L_{r-i}$$

$$= d^{r-i}$$

$$f^{-1}L_{r-i} = \underbrace{\bigcap_{j=1}^{r-i} \underbrace{f^{-1}H_j}_{\text{deg } d \text{ divisor in } \mathbb{P}^r}}_{\text{deg } d^{r-i} \text{ subvar}} \qquad \text{hyperplanes } H_j \subset \mathbb{P}^s$$

$$\Longrightarrow \quad [\Gamma_f] = \sum_{i=0}^{r} d^{r-i}\alpha^{r-i}\beta^{s-r+i}$$

in particular for $\Delta \subset \mathbb{P}^r\times\mathbb{P}^r$

$$[\Delta] = \sum_{i=0}^{r} d^{r-i}\alpha^{r-i}\beta^{s-r+i}$$

Q char(k)=0

$f: \mathbb{P}^r \longrightarrow \mathbb{P}^r$ $\quad$ deg d $\quad$ surj morph

$g: \mathbb{P}^r \longrightarrow \mathbb{P}^r$ $\quad$ deg e

In $\boxed{\text{general}}$, for how many points $x\in\mathbb{P}^r$ do we have $f(x)=g(x)$?

for general f, g : $\deg(\Delta_f \cap \Delta_g) = \sum_{i=0}^{r} d^{r-i} e^{i}$

(1) $\sigma, \gamma \in GL_r$

$$\begin{tikzcd}
\mathbb{P}^r \arrow[r, "f"] \arrow[d, "\sigma"', "\wr"] & \mathbb{P}^r \arrow[r, "\gamma"] & \mathbb{P}^r \\
\mathbb{P}^r \arrow[urr, "\gamma f \sigma^{-1}"'] & &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathbb{P}^r \times \mathbb{P}^r \arrow[r, "\sigma \times \gamma"] & \mathbb{P}^r \times \mathbb{P}^n \\
\Delta_f \arrow[u, phantom, sloped, "\subset"] \arrow[r, "\sim"] & (\sigma \times \gamma) \cdot \Delta_f = \Delta_{\gamma f \sigma^{-1}} \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

(2) $GL_r \times GL_r \circlearrowright \mathbb{P}^r \times \mathbb{P}^r$ transitively

(3) (Kleiman) $\overset{\text{alg gp}}{G} \circlearrowright X$ transitively then for every $V, W \subset X$ for gen. $g \in G$, $gV \vdash W$.

Ex $\mathbb{P}^n$, consider $V = H^0(\mathbb{P}^n, \mathcal{O}(d))$
$W = H^0(\mathbb{P}^n, \mathcal{O}(e))$

we have $V \otimes_k W \twoheadrightarrow H^0(\mathbb{P}^n, \mathcal{O}(de)) = U$

$f, g \longmapsto fg$

$$[Z(f)], [Z(g)] \longmapsto [Z(fg)]$$

$$X := \mathbb{P}(V) \times \mathbb{P}(W) \xrightarrow{\tau} \mathbb{P}U$$

induced by

$$\begin{tikzcd}
H^0(\mathbb{P}^n, \mathcal{O}(de))^\vee \arrow[d, equal] & \subset V^\vee \otimes_k W^\vee = H^0(X, pr_1^*\mathcal{O}(1) \otimes pr_2^*\mathcal{O}(1)) \arrow[d, "ev_{([Z(f)], [Z(g)])}"] \\
H^0(\mathbb{P}U, \mathcal{O}(1)) & K
\end{tikzcd}$$

$ev^\vee_{([Z(f)], [Z(g)])}$: $K \hookrightarrow U = H^0(\mathbb{P}^n, \mathcal{O}(de))$
$1 \longmapsto fg$

$H \subset \mathbb{P}^q$ hyperplane $\Rightarrow \tau^{-1}H = \underset{\text{hyperplanes}}{A \times B}$

$\Rightarrow \tau^* : A^*(\mathbb{P}^q) \longrightarrow A^*(X)$
$h \longmapsto \alpha + \beta$

Furthermore, τ is birat onto image

- $Z := \tau(\{\text{red. deg } d\} \times \mathbb{P}(W) \cup \mathbb{P}(V) \times \{\text{red deg } e\})$
  = red deg de that can be further red.
- $\tau(X) \setminus Z \overset{\text{open}}{\subset} \tau(X)$
  τ birat over it

char(k)=0 Q For general homog cubic poly $f_1, f_2, f_3 \in k[x_0, x_1, x_2]$ how many $(t_1, t_2, t_3) \in k^3$ up to scalar s.t $t_1 f_1 + t_2 f_2 + t_3 f_3$ factors nontrivially?

$$X := \overset{\text{conics}}{\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(2))} \times \overset{\text{lines}}{\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(1))} \hookrightarrow \overset{\text{cubics}}{\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))}$$

$$[Z(q)], [Z(\ell)] \longmapsto [Z(q\ell)]$$

$\Gamma := \text{im}(\tau) =$ reducible conics

codim i lin. space

$$\begin{tikzcd} L_i \subset \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3)) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{LID } f_1, \dots, f_{9-i} \in H^0(\mathbb{P}^2, \mathcal{O}(3)) \arrow[dl, leftrightarrow] \\ V_{L_i} \subset H^0(\mathbb{P}^2, \mathcal{O}(3))^\vee = H^0(\mathbb{P}^9, \mathcal{O}(1)) & \end{tikzcd}$$

$1 \le i \le 9$

dim i+1

$L_i = \{[Z(\sum t_i f_i)] \mid t_i \in k\}$
$= \mathbb{P}\langle \sum t_i \cdot f_i \rangle$
↑ formal basis

$f_1, f_2, f_3 \in H^0(\mathbb{P}^2, \mathcal{O}(3))$
general s.t $L_{f_1, f_2, f_3} :=$ dim 3 lin subspace of cubics given by $t_1 f_1 + t_2 f_2 + t_3 f_3$ $t_i \in k$
intersects Γ transversally

$$|\Gamma \cap L_{f_1, f_2, f_3}| = \deg([\Gamma] \cdot h^7) = \deg((\tau_*[X]) \cdot h^7)$$

proj form

$$= \deg((\tau^* h)^7) = \deg((\alpha + \beta)^7)$$

$$= 21$$

Parameter Spaces and Incidence-Correspondences

Ex (1) $\mathbb{P}(V)$ parametrizes lines in V

(2) $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(m))$ parametrizes deg m hypersurfaces in $\mathbb{P}^n$

(3) $\mathbb{P}\ker(H^0(\mathbb{P}^n, \mathcal{O}(m)) \xrightarrow{ev_L} H^0(L, \mathcal{O}(m)))$ parametrizes deg m hypersurfaces that contain L

(4) $G(V, d+1) = G_d(\mathbb{P}(V))$ parametrizes dim d subspaces of $\mathbb{P}(V)$

(5) $S \subset V \otimes_k \mathcal{O}_G$ $\qquad \mathbb{P}^n := \mathbb{P}(V)$

universal subbundle of rk $d+1$ on $G_d(\mathbb{P}^n)$

$L \in G$, $\quad (S^\vee)_{(L)} = H^0(L, \mathcal{O}(1))$

$\Longrightarrow$ for $m \geq 1$ $\quad (\mathrm{Sym}^m S^\vee)_{(L)} = \mathrm{Sym}^m H^0(L, \mathcal{O}(1)) = H^0(L, \mathcal{O}(m))$

$$\begin{tikzcd} \mathbb{P}\mathrm{Sym}^m S^\vee \arrow[d] \\ G \end{tikzcd}$$

param. deg m hypersurfaces in d-planes, i.e (L, H) s.t $H \subset L$

subvar

$$\mathcal{H} = \left\{ \begin{array}{l} \text{closed relation} \\ \text{between } p \in X \text{ \& } t \in T \end{array} \right\} \subset X \times T$$

Incidence-correspondence

parameter spaces

Q How many deg m hyp. in d-planes meet $r = (d+1)(n-d) + \binom{m+d}{d}$ general linear subspaces of $\mathbb{P}^n$ of dim $n-d$?

why r? $r = \dim(\mathbb{P}\mathrm{Sym}^m S^\vee)$

why $n-d$? $A \subset \mathbb{P}^n$ dim $n-d$, $A \pitchfork L \Rightarrow A \cap L = pt$
$L \subset \mathbb{P}^n$ dim d
So if $H \subset L$ deg m hyp. that meets A, then $A \cap H = pt$

$$\begin{tikzcd}
D = \{(p,L,H) \mid p \in H \subset L\} \arrow[d, phantom, "\cap"] & & & \\
X := \mathbb{P}\mathrm{Sym}^m S_I^\vee = \{(p,L,H) \mid p \in L,\ H \subset L\} \arrow[d, "\pi"] \arrow[rr, phantom, "\subset"] & & \mathbb{P}^n \times \mathbb{P}\mathrm{Sym}^m S^\vee \arrow[r, two heads] \arrow[dd] & \mathbb{P}\mathrm{Sym}^m S^\vee \arrow[d, equal] \\
 & & & \mathbb{P}\mathrm{Sym}^m pr_2^\ast S^\vee \\
I = \{(p,L) \mid p \in L\} = G_d(V \otimes_k \mathcal{O} / \mathcal{O}(-1)) = G_d(\Omega^\vee_{\mathbb{P}^n}) \arrow[rr, phantom, "\subset"] & & \mathbb{P}^n \times G =: \ast \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
 & \mathbb{P}^n & & G
\end{tikzcd}$$

$S_I := pr_2^\ast S|_I$

$\Downarrow$ Let $A \subset \mathbb{P}^n$ $(n-d)$-dim'l lin. subspace

$$\begin{tikzcd}
\rho^{-1}A = D_A := \{(p,L,H) \mid p \in A \cap H \subset L\} \arrow[d, phantom, "\cap"] \arrow[r, phantom, "\subset"] \arrow[dr, phantom, very near start, "\lrcorner"] & D \arrow[d, phantom, "\cap"] \arrow[dd, bend left=60, "\rho"] & \\
\mathbb{P}\mathrm{Sym}^m S_{I_A}^\vee \arrow[d] \arrow[r, phantom, "\subset"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{P}\mathrm{Sym}^m S_I^\vee \arrow[d] \arrow[r] & \mathbb{P}\mathrm{Sym}^m S^\vee \\
A \arrow[r, phantom, "\subset"] & \mathbb{P}^n &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{P}\mathrm{Sym}^m S_{I_A}^v \arrow[r, two heads, "f"] & \mathbb{P}\mathrm{Sym}^m S^v \\
\cup & \cup \\
D_A \arrow[r, two heads, "f_A"] & C_A := \{(L,H) \mid H \subset L,\ H \cap A \neq \emptyset\}
\end{tikzcd}$$

f, f_A birational

isom over open set of G, $L \cap A = pt$

Answer to question: $A_1, \dots, A_r \subset \mathbb{P}^n$

gen. dim $n-d$ lin. subspaces

$$\deg([C_{A_1}] \cdots [C_{A_r}])$$

$$\|$$

$$\deg([D_A] \cdots [D_A])$$

$\longleftarrow$ set $D := D_{A_1}$ $\quad$ $U = \mathrm{Dom}(f^{-1})$

$$\begin{tikzcd}
D := D_{A_1} \arrow[d, "f", "= \text{birat}"'] \\
C := C_{A_1}
\end{tikzcd}$$

We may pick $A_2, \dots, A_r$ general enough s.t

$$\begin{tikzcd}
D \cap D_{A_2} \cap \dots \cap D_{A_r} = D \cap f^{-1}A_1 \cap \dots f^{-1}A_2 \arrow[d, "f"', "\wr"] \\
C \cap C_{A_2} \cap \dots \cap C_{A_2}
\end{tikzcd}$$

pf By Kleiman-Bertini, we can pick A_{i+1} s.t $f^{-1}A_{i+1} := D_{A_{i+1}}$ reduces dim of $f^{-1}((C \cap C_{A_2} \cap .. \cap C_{A_i} \setminus U)$ by at least one. Then we get

$\dim(C \cap C_{A_2} \cap .. \cap C_{A_r} \setminus U) \leq \dim(f^{-1}((C \cap C_{A_2} \cap .. \cap C_{A_r} \setminus U) = -1$,

$C \cap C_{A_2} \cap .. \cap C_{A_r} \setminus U = \emptyset \Rightarrow C \cap C_{A_2} \cap .. \cap C_{A_r} \subseteq U$.

* Don't have tools yet, i.e chern classes. Return tomorrow ☺

$$\begin{tikzcd}
X_t \arrow[r, hook] \arrow[d, "f_t"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "f"] \\
Y_t \arrow[r, hook] \arrow[d, "p_t"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y} \arrow[d, "p"] \\
\{t\} \arrow[r, hook, "t"'] & T
\end{tikzcd}$$

var dim m

$\alpha_t := t^! \alpha$

will talk about on Day 4, for now: $D \subset \mathcal{Y}$, $c_1(\mathcal{O}_{\mathcal{Y}}(D)) \cap \alpha := \alpha|_D$

(1) E v. bd $/\mathcal{Y}$, $\alpha \in A_*(\mathcal{Y})$ $\Longrightarrow$ $c_k(E_t) \cap \alpha_t = (c_k(E) \cap \alpha)_t$, $\forall k$

(2) $p : \mathcal{Y} \longrightarrow T$ sm morph of sm var's, $\alpha, \beta \in A_*(\mathcal{Y})$ $\Longrightarrow$ $(\alpha \cdot \beta)_t = \alpha_t \cdot \beta_t$

(2) f = flat, $\alpha \in A_*(\mathcal{Y})$ $\Longrightarrow$ $(f^*\alpha)_t = f_t^* \alpha_t$

(3) f = proper, $\alpha \in A_*(\mathcal{X})$ $\Longrightarrow$ $(f_*\alpha)_t = f_{t*} \alpha_t$

EX

(1) $\mathcal{Y}$ = sm, $\downarrow p$ proper, T ← irred

$\alpha \in A_*(\mathcal{Y})$, $\beta \in A^*(\mathcal{Y})$ s.t. $\alpha \cdot \beta \in A_m(\mathcal{Y})$

$\deg(\alpha_t \cdot \beta_t) = \deg(p_{t*}(\alpha \cdot \beta)_t)$
$= \deg((p_* \alpha \cdot \beta)_t)$
$= N$ is indep of $t \in T$

$p_*(\alpha \cdot \beta) \in A_m(T)$
$\Longrightarrow p_*(\alpha \cdot \beta) = N[T]$
$\Longrightarrow p_*(\alpha \cdot \beta)_t = N[t]$

(2) $\mathcal{X} \subset T \times \mathbb{P}^n$, $\downarrow f$, T = var dim $m > 0$

if f = flat of rel dim r then call $\mathcal{X}$ a flat proj family

For any $t \in T$,

$$\begin{aligned}
\deg(X_t) &= \deg\left(c_1(\mathcal{O}(1))^r \cap [X_t]\right) \\
&= \deg\left(c_1(pr_1^*\mathcal{O}(1)|_{\mathbb{P}^n \times \{t\}})^r \cap [X_t]\right) \\
&= \deg\left(\left(c_1(pr_1^*\mathcal{O}(1))^r \cap [\mathcal{X}]\right)_t\right) \\
&= d
\end{aligned}$$

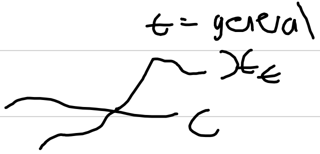
(3)

$$\begin{tikzcd} \mathcal{X} \arrow[d, "f"] \\ C \arrow[u, bend left, "s"] \end{tikzcd}$$

sm surface flat proj.

family of stable curves of genus g

sm curve



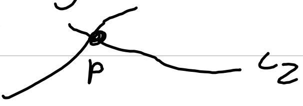
$$\begin{tikzcd} C\cap\mathcal{X}_t \arrow[r, hook] \arrow[d] & C \arrow[d, "s"] \\ \mathcal{X}_t \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] \\ t & C \arrow[l, phantom, "\in"] \end{tikzcd}$$

$[C] \in A_1(\mathcal{X})$

$[C]_t = [t] \implies \deg([C]_t) = 1$

↳ for general $t \in C$

suppose $\mathcal{X}_t = c_1$

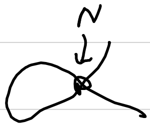


$[\mathcal{X}_t] = [c_1] + [c_2]$

$[C]_t = [C]\cdot[\mathcal{X}_t] = [C]\cdot[c_1] + [C]\cdot[c_2]$

So cannot have $p \in s(C)$

More generally, suppose $\mathcal{X}_t =$



(generically red.) nodal curve

If $|C\cap\mathcal{X}_t| = N \Rightarrow 1 = [C]_t = [C]\cdot[\mathcal{X}_t] = [C]\cdot[(\mathcal{X}_t)_{red}]$

$= [C\cap(\mathcal{X}_t)_{red}] = k\,[N]$

but $C\cap(\mathcal{X}_t)_{red} \subset (\mathcal{X}_t)_{red}$ is Cartier

$\implies k \geq 2$, an contradiction

So cannot have $N \in s(C)$

Day 4

Today
- numerical invariants $\begin{cases} \text{Chern class} \\ \text{Segre class} \end{cases}$
- deg loci and Grassmannians
  - (1) Only Grassmanians & Schubert var for V. spaces /pt
  - (2) Everything can be gen. to V. bdles/X arb scheme (see F Ch. 14)
  - (3) combinatorial exposition (uses determinantal identities) in F ch. 14
  - (4) method in (2) also constructs deg. classes that reduce to class of deg loci under certain conditions
- deg loci $\hookrightarrow$ chern classes

Chern classes

Given a V. bdle E/X of rk e, the chern classes $c_i(E)$ are unique assignment of linear maps $A_*(X) \to A_{*-i}(X)$ s.t they satisfy analogous properties to intersection products:

proj formula (a) $f: Y \to X$ proper

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*\alpha$$

flat pullback (b) $f: Y \to X$ flat

$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha)$$

commutativity (c) $c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha)$

Normalization (d) $\mathcal{L} \cong \mathcal{O}_X(D)$ $D \subset X$ Cartier

note: $c_k(E) = 0$ $\forall k > e$
$c_0(E) = 1$

$$c_1(\mathcal{L}) \cap [X] = [D]$$

Total chern class $c(E) = 1 + c_1(E) + \dots + c_e(E)$

Ex E rk e $/X$

$$\begin{tikzcd} \mathbb{P}E \arrow[d, "\pi"] \\ X \end{tikzcd}$$

$A^*(\mathbb{P}E) \overset{\text{rng isom}}{\cong} \mathbb{Z}[\xi]/\xi^{n+1}$

Where ξ is Cartier div s.t. $\mathcal{O}_{\mathbb{P}E}(\xi) = \mathcal{O}_{\mathbb{P}E}(1)$

Then $\forall \beta \in A_k(\mathbb{P}E)$, $\beta = \sum_{i=0}^{k} c_1(\mathcal{O}_1(1))^i \cap \pi^* \alpha_{k-i}$
for $\alpha_i \in A_i(X)$.

Thm (Whitney sum) $0 \to E' \to E \to E'' \to 0$
then $c(E) = c(E')\, c(E'')$.

see splitting construction, F §3.2.

Segre classes

$C = \operatorname{spec}(S^\bullet)$ $S^\bullet$ = graded $\mathcal{O}_X$-alg, f.g by S^1

e.g
- $V \subset X$ $C_V X$
- $\mathcal{E}$ loc. free sheaf on X $E = \operatorname{Spec} \operatorname{Sym}^\bullet_{\mathcal{O}_X} \mathcal{E}^\vee$

$$\begin{tikzcd} \bar{C} := \mathbb{P}(C \oplus 1) = \operatorname{Proj}(C[\overset{\text{deg } 1}{z}]) \arrow[d, "\pi"] \\ X \end{tikzcd}$$

Let $e = \dim(C)$

Define segre class of of cone

$$s(C) \cap (-) := \pi_*\left(c(\mathcal{O}_{\mathbb{P}E \oplus 1}(1))^{-1} \cap \pi^*(-)\right)$$
$$= \sum_{i=0}^{n} \pi_*\left(c_1(\mathcal{O}_{\mathbb{P}E \oplus 1}(1))^{e+i} \cap \pi^*(-)\right)$$

Def $V \subset X$ LCI segre class of subscheme $V \subset X$, $C = C_V X$

$$s(V, X) := s(C_V X) \cap [V] \in A_*(V)$$

Thm $s(E) = c(E)^{-1}$

Ex $V \subset X$ LCI $\quad S(V,X) = c(N_{V/X})^{-1} \cap [V]$

Thm

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] & \text{var} \\ V \arrow[r, hook] & Y & \text{var} \end{tikzcd}$$

(a) If f = proper & surj

$$g_* S(W,X) = \deg(X/Y)\, S(V,Y)$$

(b) If f = flat

$$g^* S(V,Y) = S(W,X)$$

Thm/def $V \subset Y$, $\quad \mathrm{mult}_V(Y) = \text{coeff of } S(V,Y)_{\dim(V)}$

Ex $p \in Y$, then $S(V,Y) = \mathrm{mult}_p Y\, [p]$

Ex $\mu : p \in X \longrightarrow \mathrm{mult}_p X$ $\quad$ $\mathrm{Char}(k)=0$, X = complete var$/k$

is constructible

Let $Z \subset X$ subvar of dim > 0

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & B = Bl_{\Gamma_Z} X \times Z \arrow[d, "\pi"] & \supset & \mathcal{Y} \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dd, bend left, "p"] \\ Z \arrow[r, hook, "\Gamma_Z"] \arrow[dr, "f"'] & X \times Z \arrow[d, "pr_2"] & \supset & X \times T \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ & Z & \supset & T \end{tikzcd}$$

Let $T \subset Z$ open s.t $\forall t \in T$

$Y_t := \pi^{-1}(X \times \{t\})$ generically equals $Bl_t X$

$$\forall t \in T,\ \mathrm{mult}_t X = \deg(S(t,X)) = \deg(\pi_* S(|E_t|, |Y_t|))$$

$$\begin{pmatrix}\text{Fulton} \\ \text{Lemma 4.2}\end{pmatrix} = \deg(\pi_*(S(E_t, Y_t)))$$

$$= \deg(\pi_*(c(E) \cap [E_t]))$$

$$= \text{Constant}$$

$$\begin{tikzcd} \tilde{B} \arrow[d, "\rho = \text{res of sing}"] \arrow[ddd, bend right, "f"'] & \\ B = Bl_{\Gamma_Z} X \times Z \arrow[d, "\pi"'] & \supset \pi^{-1}(X \times \{t\}) \arrow[d] \\ X \times Z \arrow[d, "pr_2"] & \supset X \times \{t\} = pr_2^{-1}(t) \arrow[d] \\ Z & \ni Z \end{tikzcd}$$

Let $T \subset Z$ open s.t $f^{-1}T \longrightarrow T$ is smooth and $\forall t \in T$ $\pi^{-1} pr_2^{-1}(t) \cap B_{sm} \neq \phi$.

In particular, $\forall t \in T$ $f^{-1}(t) = \rho^{-1}\pi^{-1}pr_2^{-1}(t)$ is irred & red. and $f^{-1}(t) \longrightarrow \pi^{-1}pr_2^{-1}(t)$ is birat. Since $Bl_t X = Bl_{\Gamma_Z \cap pr_2^{-1}(t)}\, pr_2^{-1}(t)$ is irred comp of $Y_t = \pi^{-1}pr_2^{-1}(t)$, we get they are generically same.

Lemma $i^![X] := s^*_{f'^*N_Y V}[C_{V'}X] = \{c(f'^*N_V Y) \cap s(V', X)\}_{n-d}$

$\Updownarrow$ equivalent $\quad \Uparrow$ blow-up

$$\begin{tikzcd} E \arrow[r, phantom, "\subset"] \arrow[d] & Bl_V Y \arrow[d] \\ V \arrow[r, phantom, "\subset"] & Y \end{tikzcd}$$

Thm (Excess Int Formula) If $i' : V' \hookrightarrow X$ is LCI

(Fulton Thm 6.3) of codim d', then

$$i^! = c_{d-d'}(f'^*N_V Y / N_{V'}X) \cap i'^!$$

Ex (1) when i' is LCI of codim $d' = d$, this is compatibility

(2)

$$\begin{tikzcd} V \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d, hook, "i"] \\ V \arrow[r, phantom, "\subset"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, equal] \\ V \arrow[r, phantom, "\subset"'] \arrow[r, phantom, "i"'] & Y \end{tikzcd}$$

$$i^* i_*[V] \overset{\text{proper}}{=} i_i^![V] = c_d(N_Y V) \cap [V]$$

self-intersection formula

(3) Recall $s : X \hookrightarrow E$ sec of v. bdle E/X of rk e

$$\begin{tikzcd} Z(s) \arrow[r, hook] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "s"] \\ X \arrow[r, hook, "s_E"'] & E \end{tikzcd}$$

$$\mathbb{Z}(s) := (s_E^!)_s^{(e)}[X]$$

$$[Z(s)] \overset{\text{if } s = \text{reg}}{=} i_* \mathbb{Z}(s) = s_E^* s_*[X] = s^* s_*[X] = \boxed{c_e(E) \cap [X]}$$

Cor Gysin commutes w/ chern classes, i.e E v. bdle / Y

$$i_f^!(c_k(E) \cap \alpha) = c_k(f^*E) \cap i_f^! \alpha$$

Ex $\mathrm{End}_{A^*(Y)}(A^*(Y)) \xrightarrow[\sim]{\cap [Y]} A^*(Y)$

Cor says that $c_k(E) \cap (-)$ is $A^*(Y)$-lin $\Rightarrow c_k(E) \in A^*(Y)$

Specifically, $c_k(E) \cap (y_1 \cdot y_2) = y_1 \cdot (c_k(E) \cap y_2)$

$$\Downarrow$$

$$\boxed{c_k(E) \cap y = (c_k(E) \cap [Y]) \cdot y}$$

Q X = sm proper $\overset{\text{sm lin alg gp}}{G} \circlearrowright X$ Let $\sigma \in G$

$k = \mathbb{C}$ If σ has fin. many fixed pts s.t the graph $\Delta_\sigma : X \hookrightarrow X \times X$ is transverse to the diagonal Δ, then there are $\chi(X^{an})$ fixed points of σ.

Condition for $\Delta \pitchfork \Delta_\sigma$:

$\forall x \in X^\sigma = \text{fixed pts of } \sigma$, $d\sigma_x$ does not have eigenvalue 1.

pf. Let $x \in \Delta \cap \Delta_\sigma$, i.e $\sigma(x) = x$

Looking at $d\Delta_\sigma(x) : T_x X \to T_x X \oplus T_x X$

$$v \mapsto v \oplus d\sigma_x(v)$$

we see $\mathrm{im}(d\Delta_\sigma(x)) \cap \mathrm{im}(d\Delta(x)) = 0$

$\Rightarrow \mathrm{im}(d\Delta_\sigma(x)) + \mathrm{im}(d\Delta(x)) = T_{(x,x)} X \times X$

$\Rightarrow \Delta \pitchfork \Delta_\sigma$ $\square$

(2.) (Kleiman transversality) $G \circlearrowright X$ transitively and $\sigma \in G$ general enough.

$$\begin{tikzcd}
& Z=\{(g,x,gx)\} \arrow[dl] \arrow[dr, two heads] & \subset G\times(X\times X) \\
G & & X\times X
\end{tikzcd}$$

Note: (1) $\forall g\in G \quad Z_g = \{g\}\times\Delta_g$

(2) $G\times X \xhookrightarrow{id\times\mu} (G\times X)\times X$ so $Z\cong G\times X$

$g, x \longmapsto (g,x), gx$ (hence $Z=sm$)

Recall: (1) $\deg([Z]_t\cdot(\pi^*\Delta)_t)$ indep of $t\in G$

(2) $\forall g\in G$, $\operatorname{codim}_Z(Z_g)=\dim(G)$. so since $Z=sm$

$Z_g\subset Z$ is LCI $\Rightarrow [Z]_g=[Z_g]=[\Delta_g]$ in $A_*(X\times X)$

$$\Longrightarrow \deg([\Delta_g]\cdot[\Delta]) = \deg([\Delta]^2)$$

$$\overset{\text{self-int form}}{=} \deg(c_n(T_X)\cap[X])$$

$$(\text{GRR + Hodge Theory}) = \chi(X^{an})$$

Q How many deg m hyp. in d-planes meet $r = (d+1)(n-d) + \binom{m+d}{d}$ general linear subspaces of $\mathbb{P}^n$ of dim $n-d$?

Recall

$A \subset \mathbb{P}^n$ $(n-d)$-dim'l lin. subspace

$$\begin{tikzcd}
D_A := \{(p,L,H) \mid p \in A \cap H \subset L\} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\lrcorner", very near start] & D = \{(p,L,H) \mid p \in H \subset L\} \arrow[d, phantom, "\cap"] & \\
X_A := \mathbb{P}\mathrm{Sym}^m S^\vee_{I_A} \arrow[r, phantom, "\subset"] \arrow[d, "\pi"] \arrow[dd, bend right, "\rho"'] \arrow[dr, phantom, "\lrcorner", very near start] & X := \mathbb{P}\mathrm{Sym}^m S^\vee_I \arrow[r] \arrow[d] & \mathbb{P}\mathrm{Sym}^m S^\vee \\
I_A \arrow[r, phantom, "\subset"] \arrow[d, "pr_1"] \arrow[dr, phantom, "\lrcorner", very near start] & I = G_d(\Omega^\vee_{\mathbb{P}^n}) \subset \mathbb{P}^n \times G \arrow[d] & \\
A \arrow[r, phantom, "\subset"] & \mathbb{P}^n &
\end{tikzcd}$$

$$\begin{tikzcd}
f: \mathbb{P}\mathrm{Sym}^\vee_{I_A} \arrow[r, two heads] & \mathbb{P}\mathrm{Sym}^m S^\vee \\
f_A: D_A \arrow[u, phantom, "\cup"] \arrow[r, two heads] & C_A := \{(H,L) \mid H \subset L,\ A \cap H \neq \emptyset\} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

f, f_A birat

$*$ D_A is closed:

on I_A

$$\begin{tikzcd}
S_{I_A} \arrow[r, hook] & V \otimes \mathcal{O}_{I_A} \\
& \rho^* \mathcal{O}(-1) \arrow[u] \arrow[ul, dashed, "e"]
\end{tikzcd}
\Longleftrightarrow$$

over $(p,L) \in I_A$

$$\begin{tikzcd}
& \Gamma(\mathfrak{m}_p(1)) \arrow[d, phantom, "\cap"] & \\
H^0(L, \mathcal{O}(1)) \arrow[dr, dashed, "ev_p"'] & H^0(\mathbb{P}^n, \mathcal{O}(1)) \arrow[l, two heads] \arrow[d, "ev_p"] & \Gamma(d_L(1)) \arrow[l, hook'] \arrow[ul, bend right, "p \in L"'] \\
& \mathcal{O}_p &
\end{tikzcd}$$

on X_A: $\mathcal{E}: \rho^*\mathcal{O}(-m) \xrightarrow{e} \pi^* \mathrm{Sym}^m S_{I_A} \longrightarrow \mathcal{O}_{X_A}(1)$

on $(p, L, H = \mathcal{Z}(g)) \in X_A$

$\mathcal{E}^\vee_{(p,L,H)}$: $k \longrightarrow H^0(L, \mathcal{O}(m)) \xrightarrow{ev_p} \mathcal{O}_p$

$1 \longmapsto g$

$\Longrightarrow$ $Z(\mathcal{E})$ is at least set-theoretically D_A

$$\begin{tikzcd}
D_A \arrow[rr, phantom, "\subseteq"] \arrow[ddrr, bend right, two heads] & & Z(\mathcal{E}) \arrow[r, phantom, "\subset"] & X_A \arrow[d, "g"] \\
& & & I_A := \{(p,L) \mid p \in L \cap A\} = Gr_d(\Omega^\vee_{\mathbb{P}^n}|_A) \arrow[d] \\
& & & A
\end{tikzcd}$$

$*$ $Z(\mathcal{E})$ is sm of codim 1 in X_A:

$\mathbb{P} := g^{-1}(p, L) = \mathbb{P}H^0(L, \mathcal{O}(m))$

$\mathcal{E}_{(p,L)}$: $\mathcal{O}_{\mathbb{P}}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}}$

$\Longrightarrow$ $g^{-1}(p, L) \cap Z(\mathcal{E}) = Z(\mathcal{E}_{(p,L)}) =$ hyperplne

$\Longrightarrow$ $\mathcal{E}$ is a reg seq & $Z(\mathcal{E})$ is CM

$$\begin{tikzcd}
g|_{Z(\mathcal{E})}: Z(\mathcal{E}) \arrow[r, two heads] & I_A \\
\text{CM} \arrow[u] & \text{smooth} \arrow[u]
\end{tikzcd}$$

$g|_{Z(\mathcal{E})}$ has sm fibers of constant dim

$\Longrightarrow$ smoothness criterion: g_A is sm morph

$\Longrightarrow$ $Z(\mathcal{E})$ is smooth

$\Longrightarrow$ $D_A = Z(\mathcal{E})$

$\Longrightarrow$ $[D_A] = c_1(\mathcal{O}_{X_A}(1) \otimes \rho^*\mathcal{O}_A(m))$

Recall For general $A_1, \ldots, A_r$

$$\begin{array}{ccc} \# | D_{A_1} \cap \ldots \cap D_{A_r} | & = & \deg([D_{A_1}] \cdot \ldots \cdot [D_{A_r}]) \\ \| & & \| \\ \# | C_{A_1} \cap \ldots \cap C_{A_r} | & & \deg(c_1(\mathcal{O}_X(1) \otimes \pi^* \mathcal{O}_A(m))^r \cap [X]) \end{array}$$

note that on $A \times G$: universal quot bdle of rk $n-d$

$$\eta : pr_1^* \mathcal{O}(-1) \to V \otimes_k \mathcal{O}_G \twoheadrightarrow Q$$

$$I_A = Z(\eta) \subset A \times G$$

$\Longrightarrow$ checking has right codim,

$$[I_A] = c_{n-d}(Q \otimes pr_1^* \mathcal{O}(1)) \cap [A \times G]$$

$$= \sum_{i=0}^{n-d} c_1(pr_1^* \mathcal{O}(1))^i c_{n-d-i}(Q)$$

Let $\xi = c_1(\mathcal{O}_X(1)) \in A_*(X_A)$

$h = c_1(pr_1^* \mathcal{O}(1)) \in A_*(I_A)$

$e = \binom{m+d}{m}$

$= rk(Sym^m S^\vee)$

$$\deg((\xi + m\pi^* h)^r \cap [X]) = \deg\left(\pi_* \sum_{i=0}^{r-e} m^{r-e-i} \binom{r}{e+i} \pi^* h^{r-e-i} \xi^{e+i} \cap [X]\right)$$

$$\overset{\text{proj-form}}{=} \deg\left(\sum_{i=0}^{r-e} m^{r-e-i} \binom{r}{e+i} h^{r-e-i} S_i(Sym^m S^\vee) \cap [I_A]\right)$$

$$= \deg\left(\sum_{i=0}^{r-e} \sum_{j=0}^{n-} m^{r-e-i} \binom{r}{e+i} h^{r-e-i+j} S_i(Sym^m S^\vee) c_{n-d-j}(Q) \cap [A \times G]\right)$$

Degeneracy Loci

$X =$ var dim n , $\sigma : \overset{\text{rk } e}{E} \to \overset{\text{rk } f}{F}$ hom. of v. bdles / X

Degeneracy Locus $D_k(\sigma) := \{ x \in X \mid \text{rk}(\sigma(x)) \leq k \} \underset{\uparrow}{=} Z(\overset{k+1}{\wedge} \sigma)$

locally given by vanishing of $(k+1)$-minors of σ.

Expected codim $(e-k)(f-k) = \text{codim}_{M_{e,f}}(\Sigma_{e,f,k})$

(where $\Sigma_{e,f,k} \subset M_{e,f}$ is determinant var of $e \times f$ matrices of rk $\leq k$)

$$\begin{tikzcd} X \arrow[r, hook, "\sigma"'] & \text{Hom}(E,F) \simeq X \times M_{e,f} \\ D_k(\sigma) \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] \arrow[ur, phantom, "\lrcorner", very near start] & X \times \Sigma_{e,f,k} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Thm There is a unique $\mathbb{D}_k(\sigma) \in A_{n-(e-k)(f-k)}(D_k(\sigma))$ s.t.

(a) If $\text{codim}_X(D_k(\sigma)) = (e-k)(f-k)$ and $X = CM$,
$$\mathbb{D}_k(\sigma) = [D_k(\sigma)]$$

(b) $f : Y \to X$, $f^*\sigma : f^*E \to f^*F$

(i) If $f =$ flat, $f^* \mathbb{D}_k(\sigma) = \mathbb{D}_k(f^*\sigma)$

(ii) If $f = LCI$, $f^! \mathbb{D}_k(\sigma) = \mathbb{D}_k(f^*\sigma)$

(iii) If $f =$ proper + $\dim(Y) = \dim(X)$,
$$f_* \mathbb{D}_k(f^*\sigma) = \deg(Y/X)\, \mathbb{D}_k(\sigma)$$

The existence of the deg class is given by existence of determinantal class :

Suppose have a flag $0 \neq A_1 \subset \dots \subset A_d \subseteq E$ $\text{rk}(A_i) = a_i$

Determinantal Locus $\Omega(\underline{A};\sigma) := \{ x \in X \mid \dim(\overbrace{\ker(\sigma(x)) \cap A_i(x)}^{\ker(\sigma : A_i \to F)}) \geq i , 1 \leq i \leq d \}$
$$= \bigcap_{i=1}^{d} Z(\overset{a_i - i + 1}{\wedge} \sigma_i)$$

locally, $\Omega(\underline{A};\sigma)$ is given by van. of $i \times i$-minors of matrix given by $\sigma_i : A_i \to F$ for all i.

Expected codim is $h = \sum \lambda_i$ where $\lambda_i = f - a_i + i$

Thm There is a unique $\Omega(\underline{A};\sigma) \in A_{n-h}(\Omega(\underline{A};\sigma))$ s.t

(a) If $\operatorname{codim}_X(\Omega(\underline{A};\sigma)) = h$ and $X = CM$, then

$$\Omega(\underline{A};\sigma) = [\Omega(\underline{A};\sigma)]$$

(b) $f : Y \to X$, $\quad f^*\sigma : f^*E \to f^*F$

$$f^*A_1 \subset \dots \subset f^*A_d \subset f^*E$$

(i) If $f =$ flat, $f^*\Omega(\underline{A};\sigma) = \Omega(f^*\underline{A}; f^*\sigma)$.

(ii) If $f =$ LCI, $f^!\Omega(\underline{A};\sigma) = \Omega(f^*\underline{A}; f^*\sigma)$.

(iii) If $f =$ proper & $\dim(Y) = \dim(X)$,

$$f_*\Omega(f^*\underline{A}; f^*\sigma) = \deg(Y/X)\,\Omega(\underline{A};\sigma)$$

There are a couple of geometric constructions of Ω, but for our enumerative purposes, give the combinatorial definition:

Thm $\Omega(\underline{A};\sigma) = \Delta_\lambda(c(F-A_1),\dots,c(F-A_d)) \cap [X]$

$\lambda = (\lambda_1 \geq \dots \geq \lambda_d \geq 0)$

Notation $c^{(1)},\dots,c^{(d)}$ Chern classes on X

$\lambda = (f \geq \lambda_1 \geq \dots \geq \lambda_d \geq 0)$ partition of f of length d

$$\Delta_\lambda(c^{(1)},\dots,c^{(d)}) := \det\begin{pmatrix} c^{(1)}_{\lambda_1} & c^{(1)}_{\lambda_1+1} & \dots & c^{(1)}_{\lambda_1+d-1} \\ \vdots & & & \vdots \\ c^{(d)}_{\lambda_d-d+1} & & \dots & c^{(d)}_{\lambda_d} \end{pmatrix}$$

$$\Delta_\lambda(c) = \Delta_\lambda(c,\dots,c)$$

$$\Delta_q^{(p)}(c) := \Delta_{(\underbrace{q,\dots,q})}(c)$$

Ex If $0 \neq A_1 \subset \dots \subset A_{e-k} \subseteq E$ $\quad rk(A_i) = k+i$

then $\Omega(\underline{A}_i ; \sigma) = D_k(\sigma)$.

$\mathbb{P}_k(\sigma) = \Omega(\underline{A}_i ; \sigma)$

Thm $\mathbb{P}_k(\sigma) = \Delta^{(e-k)}_{f-k}(c(F-E)) \cap [X])$

☆ Cor Let X = sm. suppose F v.bdle rk f/X and $s_1, \dots, s_{f-i} \in F$ sections s.t

$s : \bigoplus_{i=1}^{f-i} \mathcal{O}_X \xrightarrow{[s_1 \dots s_{f-i}]} F$

$$D_{f-(i+1)}(s) = \{x \in X \mid s_1(x), \dots, s_{f-i}(x). \text{ are dep.}\}$$

has codim $i+1$, then $c_{i+1}(F) \cap [X] = Z(\overset{f-i}{\wedge} s)$.

Ex $s \in F$ reg section $\Rightarrow c_f(F) \cap [X] = [Z(s)]$

Day 5 Recap

$\dim X = n$

E v.bdle $/X$ rk e $\rightsquigarrow$ $c(E) = 1 + c_1(E) + \dots + c_e(E)$

$c(E)^{-1} = s(E) = 1 + s_1(E) + \dots$

Recall

- $\forall \alpha \in A_m(X)$, $c_i(E) \cap \alpha \in A_{m-i}(X)$
- X = sm $\alpha \in A_*(X)$

  $c_i(E) \cap \alpha = (c_i(E) \cap [X]) \cdot \alpha$
- $0 \to E' \to E \to E'' \to 0$

  $c(E) = c(E')\,c(E'')$
- $s \in E$ s.t codim $Z(s) = 1$ and X = CM

  then $c_e(E) \cap [X] = [Z(s)]$

$\sigma : E \to F$

Given flag $A_1 \subset \dots \subset A_d \subseteq E$

Determinantal Locus $\Omega(\underline{A};\sigma) := \{x \in X \mid \dim(\overbrace{\ker(\sigma(x)) \cap A_i(x)}^{\ker(\sigma : A_i \to F)}) \geq i, 1 \leq i \leq d\}$

Let $\lambda_i = f - a_i + i$, then $(f \geq \lambda_1 \geq \dots \geq \lambda_d)$ is partion of f of length d. Expected codim is $h = \Sigma \lambda_i$

$$\Omega(\underline{A};\sigma) = \Delta_\lambda(c(F - A_1), \dots, c(F - A_d)) \cap [X] \in A_{n-h}(\Omega(\underline{A};\sigma))$$

Where $\Delta_\lambda(c^{(1)}, \dots, c^{(d)}) := \det \begin{pmatrix} c^{(1)}_{\lambda_1} & c^{(1)}_{\lambda_1+1} & \dots & c^{(1)}_{\lambda_1+d-1} \\ \vdots & & & \vdots \\ c^{(d)}_{\lambda_d-d+1} & & \dots & c^{(d)}_{\lambda_d} \end{pmatrix}$

notation: $\Delta_\lambda(c) = \Delta_\lambda(c, \dots, c)$

$\Delta^{(p)}_q(c) := \Delta_{(\underbrace{q, \dots, q})}(c)$

Main Example $\quad G = G(\overset{\text{dim } n+1}{\overset{\downarrow}{V}}, d+1) = G_d(\mathbb{P}(V))$ $\qquad \dim = (d+1)(n-d)$

$\sigma : E = V \otimes_k \mathcal{O}_G \twoheadrightarrow Q$ = universal quot bdle of dim $n-d$

Pick any flag $\mathbb{P}(V_0) \subset \dots \subset \mathbb{P}(V_d) \subset \mathbb{P}(V)$ $\qquad \dim \mathbb{P}(V_i) = a_i$

Get flag of subbdles $A_0 := V_0 \otimes_k \mathcal{O}_G \subset \dots \subset A_d := V_d \otimes_k \mathcal{O}_G \subset E$

Schubert Var $\qquad \Omega(\underline{A};\sigma) = \{ L \in G \mid \dim(L \cap \mathbb{P}(V_i)) \geq i, \ 0 \leq i \leq d \}$

$[\Omega(\underline{A})] = \Omega(\underline{A};\sigma)$ $\qquad \dim = \sum_{i=0}^{d} (a_i - i)$

$\lambda_i = n - d + i - a_i$

$c(A_i) = 1 \ \forall i \Rightarrow \Delta_\lambda(c(Q - A_0), \dots, c(Q - A_d)) = \Delta_\lambda(c(Q))$

$\Rightarrow [\Omega(\underline{A};\sigma)] = \Delta_\lambda(c(Q)) \cap [G]$

Note: indep of $\underline{A}$ $\quad \left(= \det(\sigma_{\lambda_i + j - i})_{0 \leq i,j \leq d} \cap [G] \right)$

Let λ be partition of $n-d$ of length $d+1$

$\{\lambda\} = \{\lambda_0, \dots, \lambda_d\} := \Delta_{(\lambda_0 \geq \dots \geq \lambda_d)}(c(Q))$

$0 \leq a_0 \leq \dots \leq a_d \leq n$ $\qquad (a_0, \dots, a_d) := \Delta_{(n-d-a_0 \geq \dots \geq n-d-(d-a_d))}(c(Q)) \cap [G]$

Schubert Cycle

Ex (1) $\{0, \dots, 0\} \cap [G] = (n-d, n-d+1, \dots, n) = [G]$

$L = V_d$ $\quad$ (2) $\{n-d, \dots, n-d\} \cap [G] = (0, 1, \dots, d) = [pt]$.

(3) $\forall m$ $\qquad \sigma_m := c_m(Q) = \{m, 0, \dots, 0\}$

$\sigma_m \cap [G] = (n-d-m, n-d+1, \dots, n)$

$A \subset \mathbb{P}(V)$ of codim $d+m$

$\Sigma^{n,d}(A) = \{ L \in G \mid L \cap A \neq \emptyset \}$

then $\sigma_m \cap [G] = [\Sigma^{n,d}(A)]$

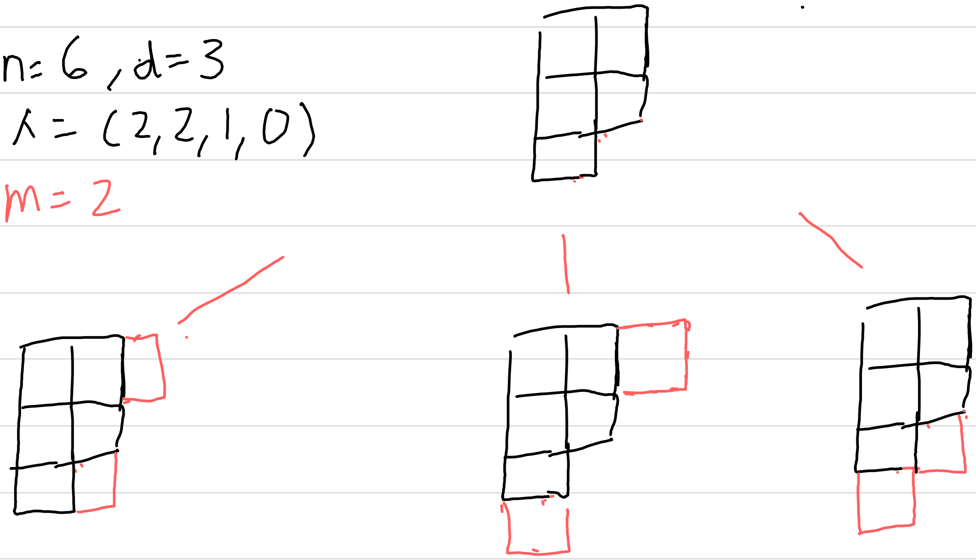
Thm $$A^k(G) = \bigoplus_{\substack{\lambda=(n-d\geq\lambda_1\geq\dots\geq\lambda_d\geq 0)\\ |\lambda|:=\sum\lambda_i = k}} \mathbb{Z}\cdot\{\lambda\}$$

(Pieri's Formula) • $$\{\lambda\}\cdot\sigma_m = \sum_{\substack{\sum\mu=\sum\lambda+m\\ \mu_i\geq\lambda_i\geq\mu_{i+1}}} \{\mu\}$$

Ex $n=6,\ d=3$

$\lambda=(2,2,1,0)$

$m=2$



(duality Thm) • $|\lambda|+|\mu|=(d+1)(n-d)$

$$\{\lambda\}\cdot\{\mu\} = \begin{cases} [pt] & \lambda_i+\mu_{d-i}=n-d\ \forall i \\ 0 & \text{else} \end{cases}$$

• $$\Delta_\lambda\cdot\Delta_\mu = \sum_{|Q|=|\lambda|+|\mu|} N_{\lambda,\mu,Q}\,\Delta_Q$$

$N_{\lambda,\mu,Q}$ given by Littlewood-Richardson rule

Ex $G = G(4,2) = G_1(\mathbb{P}^3)$ dim 4

$A^0(G)$	$\{0,0\}$	$(2,3)=[G]$	$A_4(G)$
$A^1(G)$	$\{1,0\}=\sigma_1$	$(1,3)$	$A_3(G)$
$A^2(G)$	$\{2,0\}=\sigma_2$	$(0,3)$	$A_2(G)$
	$\{1,1\}$	$(1,2)$	
$A^3(G)$	$\{2,1\}$	$(0,2)$	$A_1(G)$
$A^4(G)$	$\{2,2\}$	$(0,1)=[pt]$	$A_0(G)$

Q How many lines in $\mathbb{P}^3$ meet 4 given lines in general position?

A_1, A_2, A_3, A_4 general lines (i.e codim 2)

$$\Sigma^{3,1}(A_i) := \{L \in G_1(\mathbb{P}^3) \mid L \cap A_i \neq \emptyset\}$$

Recall $[\Sigma^{3,1}(A_i)] = \sigma_1$

$$\Rightarrow |\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 \cap \Sigma_4| \overset{\substack{A_i \text{ are gen.}\\ \downarrow}}{=} \deg(\sigma_1^4 \cap [G])$$

$$\sigma_1^2 = \{1,0\}\cdot\sigma_1 = \sigma_2 + \{1,1\}$$

$$\text{duality Thm} \Rightarrow \begin{cases} \sigma_2^2 = \{1,1\}^2 = \{2,2\} \\ \{1,1\}\cdot\sigma_2 = 0 \end{cases}$$

$$\Rightarrow \sigma_1^4 = 2\{2,2\}$$

$$\Rightarrow \sigma_1^4 \cap [G] = 2[pt]$$

So these are 2 such lines.

Fano scheme $\mathbb{P}^n := \mathbb{P}(V)$ V dim n v. space

$L = \mathbb{P}(W)$ $W \subset V$ dim $d+1$ subspace

(*) $g \in H^0(\mathbb{P}^n, \mathcal{O}(m)) = \mathrm{Sym}^m V^\vee \overset{ev_L}{\twoheadrightarrow} H^0(L, \mathcal{O}_L(m)) = \mathrm{Sym}^m W^\vee$

$$\ker(ev_L) = \Gamma(\mathbb{P}^n, \mathcal{I}_L^m)$$

$$ev_L(g) = 0 \iff L \subset Z(g)$$

on $G = G_d(\mathbb{P}^n)$ $E \hookrightarrow \mathrm{Sym}^m V^\vee \otimes_k \mathcal{O}_G \twoheadrightarrow \mathrm{Sym}^m S^\vee$

$$\begin{tikzcd}
 & F_d = \mathbb{P}E \arrow[d, phantom, "\cap"] \arrow[ddl, bend right, "g"'] \arrow[ddr, bend left, "f"] & \\
 & \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(m)) \times G_d(\mathbb{P}^n) = \mathbb{P}\mathrm{Sym}^m V^\vee \otimes_k \mathcal{O}_G \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(m)) & & G = G_d(\mathbb{P}^n)
\end{tikzcd}$$

$\beta_d : pr_1^*\mathcal{O}(-m) \to \mathrm{Sym}^m V^\vee \otimes_k \mathcal{O}_X \twoheadrightarrow pr_2^* \mathrm{Sym}^m S^\vee$

over $(Z(g), L) \in X$

$$\begin{tikzcd}
k \arrow[r, hook] & H^0(\mathbb{P}^n, \mathcal{O}(m)) \arrow[r, "ev_L"'] & H^0(L, \mathcal{O}_L(m)) \arrow[d, equal] \\
1 \arrow[r, mapsto] & g & (\mathrm{Sym}^m S^\vee)_{(L)}
\end{tikzcd}$$

Call F_d universal Fano scheme

Claim $F_d = Z(\beta_d)$

pf checking β_d at every $(x, L) \in \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(m)) \times G$, $(\beta_d)_{(x,L)} = 0$ iff $x \subset L$ from (*)

So set-theoretically are same.

Over $L \in G$, $f^{-1}([L]) = \mathbb{P}\ker(ev_L) \subset \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(m)) =: \mathbb{P}$

$$\mathcal{E} := \mathfrak{f}_d\big|_{pr_2^{-1}(L)}^{\vee} : \mathcal{O}_{\mathbb{P}}(m) \twoheadleftarrow H^0(\mathbb{P}^n; \mathcal{O}(m))^{\vee} \otimes \mathcal{O}_{\mathbb{P}} \overset{ev_L^{\vee}}{\supset} H^0(L, \mathcal{O}_L(m))^{\vee} \otimes \mathcal{O}_{\mathbb{P}}$$

$$\implies \Gamma(\mathcal{O}_{\mathbb{P}}(1)) = H^0(\mathbb{P}^n, \mathcal{O}(m))^{\vee} \supset \Gamma(\mathcal{I}_{Z(\mathcal{E})}(1)) = H^0(L, \mathcal{O}(m)) = \Gamma(\mathcal{I}_{\mathbb{P}\ker(ev_L)}(1))$$

$\implies$ $\mathcal{I}_{Z(\mathcal{E})} \subseteq \mathcal{I}_{\mathbb{P}\ker(ev_L)}$ is equality

so $f^{-1}([L]) = Z(\mathcal{E})$ = fibers of $Z(\mathfrak{f}_d)$ over $L \in G$

is sm fiber of $\binom{m+n}{m} - \binom{m+d}{m}$

$\implies$ $Z(\mathfrak{f}_d) \subset \mathcal{F}$ is CM and $Z(\mathfrak{f}_d) \to G$ has smooth fibers so $Z(\mathfrak{f}_d) = F_d$ = sm

In particular, $\mathfrak{f}_d$ is a reg section

i.e $[F_d] = c_{\binom{m+d}{m}}(pr_2^* Sym^m S^{\vee} \otimes pr_1^* \mathcal{O}(m))$

For a general hyp of deg m $X \in \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(m))$

by Kleiman-Bertini, $F_d \pitchfork pr_1^{-1}([X])$

$$\implies g^{-1}([X]) = F_d \cap pr_1^{-1}([X]) = \{L \in G \mid L \subset X\}$$

and $[F_d \cap pr_1^{-1}([X])] = [F_d] \cdot [\{[X]\} \times G] = c_{\binom{m+d}{m}}(Sym^m S^{\vee}) \cap [G]$

<u>Ex</u> $d = 1$ $\dim G_1(\mathbb{P}^n) = 2(n-1)$

$m \geq 2(n-1)$: general deg m hyp. have no lines

$m = 2n-3$: general hyp. of deg $2n-3$ have finitely many lines... how many???

Recall: $\sigma_k = c_k(Q)$

$\nearrow$ class in $A^*(G_d(\mathbb{P}^n))$ of d-dim'l lin. subspaces which meet a given $(d+k)$-codim'l lin. subspace

$\Longrightarrow$ $c(S) = (1+\sigma_1+\dots+\sigma_{n-d})^{-1}$ by whitney sum

$d=1$ S has rk 2

$$c(S) = (1+\sigma_1+\sigma_2+\dots+\sigma_{n-1})^{-1}$$
$$= 1-\sigma_1+\sigma_1^2-\sigma_2$$

$$= 1-\sigma_1+(\sigma_2+\{1,1\})-\sigma_2$$
$$= 1-\sigma_1+\{1,1\}$$
$$\Longrightarrow c(S^\vee) = 1+\sigma_1+\{1,1\}$$

Splitting principle trick: pretend $S^\vee = L_1 \oplus L_2$ $\Rightarrow c(S^\vee) = (1+\alpha_1)(1+\alpha_2$

(can do this by embedding in large enough chow ring)

$$\Longrightarrow c(S^\vee) = (1+\alpha_1)(1+\alpha_2)$$
$$\alpha_1+\alpha_2 = \sigma_1$$
$$\alpha_1 \cdot \alpha_2 = \{1,1\}$$

$$\mathrm{Sym}^m S^\vee = \bigoplus_{i=0}^{m} L^i M^{m-i}$$

$$c(\mathrm{Sym}^3 S^\vee) = \prod_{i=0}^{m}(1+i\alpha+(m-i)\beta)$$

Ex $m=3$

$$c_4(\mathrm{Sym}^3 S^\vee) = 3\beta(\alpha+2\beta)(2\alpha+\beta)3\alpha$$
$$= 9\alpha\beta(2\alpha^2+2\beta^2+5\alpha\beta)$$
$$= 9\alpha\beta(2(\alpha+\beta)^2+\alpha\beta)$$
$$= 18\{1,1\}\cdot\sigma_1^2 + 9\{1,1\}^2$$
$$\text{(duality Thm)} = 18[pt] + 9[pt]$$

$n>3$: can compute other combinatorial info with σ_k classes.

$n=3$: cubic surfaces in $\mathbb{P}^3$ have 27 lines.