Local cohomology modules over a regular ring are known in many cases to exhibit remarkable finiteness properties. For a regular ring $S$, one may ask whether the set $\operatorname{Ass} H^i_I(S)$ is finite for any ideal $I \subseteq S$ and any $i \geq 0$. A celebrated theorem originally due to Huneke and Sharp [HS93] (later generalized by Lyubeznik’s theory of $F$-modules [Lyu97]) says that this is indeed the case if $S$ has prime characteristic $p > 0$. Lyubeznik proved [Lyu93] the corresponding statement for regular local rings containing a field of characteristic 0. Concerning regular rings of mixed characteristic, the property is known to hold when $S$ is an unramified regular local ring [Lyu00], a smooth $\mathbb{Z}$-algebra [BBLSZ14], or is local and of dimension $\leq 4$ [Mar01].

The finiteness properties of local cohomology modules $H^i_I(S)$ when $S$ is a complete intersection are far less well-understood than when $S$ is regular. Infinite sets of associated primes can be found even when $S$ is a hypersurface. For example, Singh [Sin00] describes a hypersurface $S$ finitely generated over $\mathbb{Z}$,

$$S = \mathbb{Z}[u, v, w, x, y, z] / \frac{wx + vy + wz}{ux + vy + wz}$$

where $H^3_{(x,y,z)}(S)$ has a nonzero $p$-torsion element for all prime integers $p$. This is not just a global phenomenon: Katzman [Kat02] later found a local hypersurface $S$ containing a field $K$,

$$S = \frac{K[[u, v, w, x, y, z]]}{wu^2x^2 - (w + z)uxvy + zv^2y^2}$$

such that $\operatorname{Ass} H^2_{(u,v)}(S)$ is infinite. Singh and Swanson [SS04] construct similar examples of equicharacteristic local hypersurfaces and demonstrate that $\operatorname{Ass} H^i_I(S)$ can be infinite even if $S$ is a UFD, an $F$-regular ring of characteristic $p > 0$, or a characteristic 0 ring with rational
singularities. Surprisingly, at least in characteristic $p$, the local cohomology of a hypersurface still exhibits striking finiteness properties. A result proved independently by Katzman and Zhang [KZ17] or by Hochster and Núñez-Betancourt [HNB17] says that if $R$ is a regular ring of prime characteristic $p$, $f \in R$ a nonzerodivisor, and $S = R/f$, then for any ideal $I$ and any $i \geq 0$, $H^i_I(S)$ has only finitely many minimal associated primes. Equivalently, $\text{Supp} \, H^i_I(S)$ is a Zariski closed set. Other known cases of local cohomology modules with closed support include when $S$ has dimension at most 4, or when $I$ has cohomological dimension at most 2 [HKM09].

A motivating question for this paper is whether the hypersurface result generalizes to complete intersections of higher codimension.

**Question 1a.** Let $S$ be a complete intersection, let $I$ be an arbitrary ideal of $S$, and fix $i \geq 0$. Is $\text{Supp} \, H^i_I(S)$ a Zariski closed set?

For simplicity, we restrict our focus to when $S$ is given by a presentation $S = R/J$ for $R$ a regular ring, and $J$ an ideal generated by a regular sequence in $R$. Not all regular rings $R$ are known to have the property that $\text{Ass} \, H^i_I(R)$ is always finite. In order to avoid potential complications arising from $R$, we impose an additional hypothesis.

**Definition 1.1.** Call a Noetherian ring $R$ **LC-finite** if it has the property that for any ideal $I \subseteq R$ and any $i \geq 0$, the module $H^i_I(R)$ has finitely many associated primes.

We do not assume regularity in the definition. A semilocal ring of dimension at most 1 is trivially LC-finite, but can easily fail to be regular. The class of LC-finite regular rings includes, for example, all regular rings of dimension $\leq 4$, all regular rings of prime characteristic $p$, regular local rings containing a field of characteristic 0, unramified regular local rings of mixed characteristic, and smooth $\mathbb{Z}$-algebras. The class of LC-finite rings is closed under localization. If there is a finite set of maximal ideals $m_1, \ldots, m_t$ of $R$ such that $\text{Spec}(R) - \{m_1, \ldots, m_t\}$ can be covered by finitely many charts $\text{Spec}(R_f)$, each of which is LC-finite, then $R$ is LC-finite. For example, a ring of prime characteristic $p$ with isolated singularities is LC-finite. If $R$ is LC-finite and $A \to R$ is pure (e.g., if $A$ is a direct summand of $R$), then $A$ is LC-finite (see e.g., 3.1(d) in [HNB17]).

**Question 1b.** Let $R$ be an LC-finite regular ring, $J \subseteq R$ be an ideal generated by a regular sequence, and $S = R/J$. Let $I$ be an arbitrary ideal of $S$, and fix $i \geq 0$. Is $\text{Supp} \, H^i_I(S)$ a Zariski closed set?

If $R$ is a regular ring of prime characteristic $p$, Hochster and Núñez-Betancourt [HNB17] show that if $\text{Ass} \, H^{i+1}_I(J)$ is finite, then $\text{Supp} \, H^i_I(R/J)$ is closed. This raises a very natural question. Although (to the best of our knowledge) it is not yet known whether Hochster and Núñez-Betancourt’s result generalizes outside of characteristic $p$, we will nonetheless pose the question for LC-finite regular rings in any characteristic.

**Question 2.** Let $R$ be an LC-finite regular ring, $J \subseteq R$ be an ideal generated by a regular sequence, $I$ be an ideal of $R$ containing $J$ (corresponding to an arbitrary ideal of $R/J$), and fix $i \geq 0$. Is $\text{Ass} \, H^i_J(J)$ a finite set?

In Section 2, using some properties of the ideal transform functor associated with $I$, we show that this question has a positive answer when $i = 2$. In fact, we don’t need the full list of hypotheses on $R$ and $J$.

**Theorem (2.1).** Let $R$ be a locally almost factorial (see Definition 2.1) Noetherian normal ring, and let $I$, $J$ be ideals of $R$. The set $\text{Ass} \, H^2_J(J)$ is finite.
The result does not generalize to \( i > 2 \). In Section 3, we give an example where \( H^2_i(J) \) has infinitely many associated primes. In this example, \( R \) is a 7-dimensional formal power series ring, \( J \) is generated by a regular sequence of length 2, and \( I \) is 4-generated. Question 2 therefore has a negative answer at the level of generality in which it is stated. However, our counterexample crucially requires \( \text{Ass} \ H^2_i(R/J) \) to be infinite. In fact, \( R/J \) in our counterexample is precisely Katzman’s hypersurface. The nature of the counterexample therefore begs the following natural question, that, to the best of our knowledge, remains open.

**Question 3.** Let \( R \) be an LC-finite regular ring, \( J \subseteq R \) be an ideal generated by a regular sequence, \( I \) be an ideal of \( R \) containing \( J \), and fix \( i \geq 0 \). Does the finiteness of \( \text{Ass} \ H^{i-1}_I(R/J) \) imply the finiteness of \( \text{Ass} \ H^i_I(J) \)?

We investigate this question in Section 4 for various small values of \( i \). We give a partial positive answer when \( i \leq 4 \).

Notice that if \( \text{depth}_j(R) = 1 \), then \( J \simeq R \) as an \( R \)-module, so the question has a trivially positive answer. The question is only interesting when \( \text{depth}_j(R) \geq 2 \), and our methods in fact require \( \text{depth}_j(R) \geq 2 \).

**Theorem (4.1).** Let \( R \) be an LC-finite regular ring, let \( J \subseteq R \) be an ideal generated generated by a regular sequence of length \( j \geq 2 \), and let \( S = R/J \). For \( I \) an ideal of \( R \) such that \( I \supseteq J \),

(i) \( \text{Ass} \ H^i_I(J) \) and \( \text{Ass} \ H^{i-1}_I(S) \) are always finite for \( i \leq 2 \).

(ii) If the irreducible components of \( \text{Spec}(S) \) are disjoint (e.g. \( S \) is a domain), then \( \text{Ass} \ H^3_I(J) \) is finite if and only if \( \text{Ass} \ H^3_I(S) \) is finite.

(iii) If \( S \) is normal and locally almost factorial (e.g. \( S \) is a UFD), then \( \text{Ass} \ H^4_I(J) \) is finite if and only if \( \text{Ass} \ H^3_I(S) \) is finite.

We emphasize in this result that \( R \) is allowed to have any characteristic, even mixed characteristic, so long as it satisfies the LC-finiteness condition.

Given that our control over the associated primes of \( H^i_I(J) \) tends to become greater as our hypotheses on \( R/J \) become more restrictive, we turn in Section 5 to investigating the extreme case in which \( R/J \) is regular and LC-finite. At least in characteristic \( p > 0 \), relying heavily on Lyubeznik’s theory of \( F \)-modules [Lyu97], we obtain the following result.

**Theorem (5.2).** Let \( R \) be a regular ring of prime characteristic \( p > 0 \), let \( J \subseteq R \) be an ideal such that \( R/J \) is regular, and let \( I \) be any ideal of \( R \). For all \( i \geq 0 \), \( \text{Ass} \ H^i_I(J) \) is finite.

**Convention:** Throughout this paper, we assume that all given rings are Noetherian, unless stated otherwise.

## 2. Finiteness of \( \text{Ass} \ H^2_I(J) \)

If we are focused exclusively on cohomological degree 2, many of the hypotheses of our basic setting can be relaxed. \( R \) need only be normal with a condition somewhat weaker than local factoriality, and the ideal \( J \subseteq R \) can be completely arbitrary – we do not need to \( J \) to be generated by a regular sequence. Our goal is to show that \( \text{Ass} \ H^2_I(J) \) is finite for any ideal \( I \subseteq R \). The main case is when \( \text{depth}_i(R) = 1 \).

**Lemma 2.1.** Let \( R \) be a Noetherian domain, and let \( J \subseteq R \) be an ideal. If \( I \subseteq R \) is an ideal such that \( \text{depth}_i(R) \neq 1 \), then \( \text{Ass} \ H^2_I(J) \) is finite.
Proof. If $I = (0)$ or $I = R$, the result is obvious, and so we assume $I$ is a nonzero proper ideal. Since $R$ is a domain, this implies that both $J$ and $R$ are $I$-torsionfree, giving $\text{depth}_I(R) > 0$ and by hypothesis $\text{depth}_I(R) \neq 1$, so we have $\text{depth}_I(R) \geq 2$. The following sequence is exact.

$$
\begin{array}{cccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow \\
& & \downarrow & & \downarrow & \\
& & H^1_J(R/J) & \rightarrow & 0 & \rightarrow \\
& & \downarrow & & \downarrow & \\
& & H^2_J(R) & \rightarrow & H^2_J(R/J) & \\
\end{array}
$$

Note that $H^1_J(R) \cong \Gamma_I(R/J)$ is finitely generated, meaning that $H^2_J(R)$ is either finitely generated or the first non-finitely-generated local cohomology module of $J$ on $I$, and the stated result follows [BL00].

\begin{lemma}
Let $R$ be a Noetherian normal ring, and $I \subseteq R$ be an ideal such that $\text{depth}_I(R) = 1$. Then $\sqrt{I} = L \cap I_0$ for some ideal $L$ given by the intersection of height one primes, and some ideal $I_0 \subseteq R$ with $\text{depth}_{I_0}(R) \geq 2$.
\end{lemma}

Proof. In a Noetherian normal ring $R$, $\text{ht}(I_0) \geq 2$ implies $\text{depth}_{I_0}(R) \geq 2$.

Recall the notion of an almost factorial ring.

\begin{definition}
A normal domain $R$ is called \textit{almost factorial} if the class group of $R$ is torsion. A normal ring $R$ is called \textit{locally almost factorial} if $R_P$ is almost factorial for all $P \in \text{Spec}(R)$.
\end{definition}

A regular ring, for example, is locally (almost) factorial. See Hellus [Hel01] for some results on the local cohomology of almost factorial rings. Hellus shows that an almost factorial Cohen-Macaulay local ring of dimension at most four is LC-finite. Our application of the almost factorial hypothesis is motivated by the application in [Hel01].

The main property we require is that every height 1 prime of an almost factorial ring is principal up to taking radicals. Recall that an ideal $I$ is said to have pure height $h$ if every minimal prime of $I$ has height exactly $h$. An ideal of pure height 1 is (up to radical) a product of height 1 primes, so in an almost factorial ring, any pure height 1 ideal is principal up to radicals.

\begin{lemma}
Let $R$ be a locally almost factorial Noetherian normal ring, and $L$ be an ideal of pure height 1. Then there is a finite cover of $\text{Spec}(R)$ by charts $\text{Spec}(R_{f_1}), \ldots, \text{Spec}(R_{f_t})$ such that for each $i$, $LR_{f_i}$ has the same radical as a principal ideal.
\end{lemma}

Proof. We do no harm in replacing $L$ with $\sqrt{L}$, so assume $L$ is radical. Consider a single point $P \in \text{Spec}(R)$. Since $R_P$ is almost factorial, we can write $LR_P = yR_P$ for some $y \in R_P$. Up to multiplying by units of $R_P$, we may assume that $y$ is an element of $R$. Since $y \in LR_P \cap R$, there is some $u \in R - P$ such that $uy \in L$. Also, since $R$ is Noetherian, there is some $n > 0$ such that $L^nR_P \subseteq yR_P$, hence $L^n \subseteq yR_P \cap R$, and there is some $v \in R - P$ such that $vL^n \subseteq yR$. If $f = uv$, then we see that $y \in LR_f$ and $L^n \subseteq yR_f$, giving $LR_f = yR_f$.

Our choice of $f$ depends on $P$. Varying over all $P \in \text{Spec}(R)$, we obtain a collection of open charts $\{\text{Spec}(R_{f_P})\}_{P \in \text{Spec}(R)}$ which cover $\text{Spec}(R)$ such that (the expansion of) $L$ is principal up to radicals on each chart. Since $\text{Spec}(R)$ is quasicompact, finitely many of these charts cover the whole space. \hfill \Box
Corollary 2.1. Let $R$ be a locally almost factorial Noetherian normal ring, and $I \subseteq R$ be an ideal such that $\text{depth}_I(R) = 1$. Then there is an ideal $I_0 \subseteq R$ with $\text{depth}_{I_0}(R) \geq 2$, and a finite cover of $\text{Spec}(R)$ by open charts $\text{Spec}(R_{f_1}), \ldots, \text{Spec}(R_{f_i})$ such that for each $i$, $y_iR_{f_i} \cap I_0$ for some some $y_i \in R$.

To show that $\text{Ass} H^2_I(J)$ is finite, it would certainly suffice to show that $\text{Ass} H^2_{I_{R_f}}(JR_f)$ is finite on each chart $\text{Spec}(R_f)$ of a finite cover of $\text{Spec}(R)$. Thus, up to radicals, the main case to understand is when $I$ has the form $yR \cap I_0$, with $y \in R$ and $\text{depth}_{I_0}(R) \geq 2$. This decomposition can be used to get a better understanding of the $I$-transform functor.

For an ideal $\mathfrak{a} \subseteq R$, recall the $\mathfrak{a}$-transform

$$D_\mathfrak{a}(-) := \lim_{\to} \text{Hom}_R(\mathfrak{a}^t, -)$$

is a left exact covariant functor whose right derived functors satisfy $\mathcal{R}^i D_\mathfrak{a}(-) \simeq H^{i+1}_\mathfrak{a}(-)$. There is a sense in which $D_\mathfrak{a}(-)$ forces $\text{depth}_\mathfrak{a}(-) \geq 2$ without modifying higher local cohomology on $\mathfrak{a}$. Namely, for any $R$-module $M$, $\Gamma_\mathfrak{a}(D_\mathfrak{a}(M)) = H^1_\mathfrak{a}(D_\mathfrak{a}(M)) = 0$, and $H^i_\mathfrak{a}(D_\mathfrak{a}(M)) = H^i_\mathfrak{a}(M)$ for all $i \geq 2$. We require a few fundamental facts about the ideal transform, all of which can be found in the treatment presented in Chapter 2 of Brodmann & Sharp [BS12], along with proofs of the previous assertions.

Notational remark: If $F$ and $G$ are functors $\mathcal{C} \to \mathcal{D}$, we will write natural transformations from $F$ to $G$ as $\phi(-) : F(-) \to G(-)$, which consists of the data of a map denoted $\eta(A) : F(A) \to G(A)$ for each object $A$ of $\mathcal{C}$, such that for any $\mathcal{C}$-morphism $f : A \to B$, $\eta(B) \circ F(f) = G(f) \circ \eta(A)$.

Lemma 2.4. [Brodmann & Sharp [BS12], 2.2.4(i)] Let $R$ be a Noetherian ring and fix an ideal $\mathfrak{a} \subseteq R$. There is a natural transformation $\eta_\mathfrak{a}(-) : \text{Id} \to D_\mathfrak{a}(-)$ such that, for any $R$-module $M$, there is an exact sequence

$$0 \to \Gamma_\mathfrak{a}(M) \to M \xrightarrow{\eta_\mathfrak{a}(M)} D_\mathfrak{a}(M) \to H^1_\mathfrak{a}(M) \to 0$$

Lemma 2.5. [Brodmann & Sharp [BS12], 2.2.13] Let $R$ be a Noetherian ring, and $\mathfrak{a} \subseteq R$ be an ideal. Let $e : M \to M'$ be a homomorphism of $R$-modules such that $\text{Ker}(e)$ and $\text{Coker}(e)$ are both $\mathfrak{a}$-torsion. Then

(i) The map $D_\mathfrak{a}(e) : D_\mathfrak{a}(M) \to D_\mathfrak{a}(M')$ is an isomorphism.

(ii) There is a unique $R$-module homomorphism $\varphi : M' \to D_\mathfrak{a}(M)$ such that the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{e} & M' \\
\downarrow{\eta_M} & & \downarrow{\varphi} \\
D_\mathfrak{a}(M) & &
\end{array}
$$

commutes. In fact, $\varphi = D_\mathfrak{a}(e)^{-1} \circ \eta_\mathfrak{a}(M')$.

(iii) The map $\psi'$ of (ii) is an isomorphism if and only if $\eta_\mathfrak{a}(M')$ is an isomorphism, and this is the case if and only if $\Gamma_\mathfrak{a}(M') = H^1_\mathfrak{a}(M') = 0$.

The main ingredient to dealing with $H^2_I(J)$ when $\text{depth}_I(R) = 1$ is the following compatibility property of the $I$-transform functor with our decomposition $I = yR \cap I_0$.

Proposition 2.1. Let $R$ be a Noetherian ring, $y$ an element of $R$, and $I_0 \subseteq R$ an ideal. Let $I = yR \cap I_0$. There is a natural isomorphism of functors $D_{I_0}(-)_y \simeq D_I(-)$.

Proof. Precomposing $\eta_{I_0}(-)_y : (-)_y \to D_{I_0}(-)_y$ with $\text{Id} \to (-)_y$ we obtain a natural transformation $\gamma(-) : \text{Id} \to D_{I_0}(-)_y$. We claim that for any module $M$, both the kernel and cokernel of
\[ \gamma(M) : M \to D_{I_0}(M)_y \] are \( I = yR \cap I_0 \)-torsion:

- \( \ker(\gamma(M)) \) consists of those \( m \in M \) such that \( m/1 \in \Gamma_{I_0}(M)_y \), or, equivalently, \( y^t m \in \Gamma_{I_0}(M) \) for some \( t \geq 0 \). Let \( s \) be such that \( I_0^s y^t m = 0 \). Then \( (yI_0)^{\max(s,t)} = 0 \), so \( m \in \Gamma_{yI_0}(M) = \Gamma_{yR \cap I_0}(M) \) (since \( \sqrt{yI_0} = \sqrt{yR \cap I_0} \)).

- An element of \( C = \text{Coker}(\gamma(M)) \) can be represented by \( c = f/y^t \) for some \( f \in D_{I_0}(M) \), \( t \geq 0 \). Coker \( (\eta_{I_0}(M)) \) is \( I_0 \)-torsion, so there is some \( s \) such that \( I_0^s f \subseteq \text{Im}(\eta_{I_0}(M)) \). Since \( f = y^t c \), we have \( (yI_0)^{\max(s,t)} c \subseteq \text{Im}(\gamma(M)) \). The element of \( C \) represented by \( c \) therefore belongs to \( \Gamma_{yI_0}(C) = \Gamma_{yR \cap I_0}(C) \).

**Lemma 2.5**(ii) therefore gives a map \( \varphi(M) : D_{I_0}(M)_y \to D_I(M) \), specifically \( \varphi(M) = D_I(\gamma(M))^{-1} \circ \eta_I(D_{I_0}(M)_y) \). Both of the composite maps come from natural transformations \( D_I(\gamma(-))^{-1} \) and \( \eta_I(D_I(-)_y) \), so the result is a natural transformation \( \varphi(-) : D_{I_0}(-)_y \to D_I(-) \).

It remains to show that that \( \varphi(M) \) is an isomorphism for each \( M \), which is equivalent, by **Lemma 2.5**(iii), to showing that \( \Gamma_I(D_{I_0}(M)_y) = H^1_I(D_{I_0}(M)_y) = 0 \). This can be done using the Mayer-Vietoris sequence associated with the intersection \( yR \cap I_0 \). Each module \( H^i_{yR \cap I_0}(D_{I_0}(M)_y) \) vanishes because \( y \in yR + I_0 \) acts as a unit on \( D_{I_0}(M)_y \) and likewise for the modules \( \Gamma_{yR}(D_{I_0}(M)_y) \) and \( H^1_{yR}(D_{I_0}(M)_y) \). Note that \( \text{depth}_{I_0}(D_{I_0}(M)) \geq 2 \), and localization can only make depth go up, so, \( \Gamma_{I_0}(D_{I_0}(M)_y) = H^1_{I_0}(D_{I_0}(M)_y) = 0 \).

We can now see that \( \Gamma_I(D_{I_0}(M)_y) = H^1_I(D_{I_0}(M)_y) = 0 \), as desired. \( \square \)

**Corollary 2.2.** Let \( R \) be a Noetherian ring, \( y \) an element of \( R \), and \( I_0 \subseteq R \) an ideal. Let \( I = yR \cap I_0 \). Then for all \( i \geq 2 \), there is a natural isomorphism of functors \( H^i_I(-) \approx H^i_{I_0}(-)_y \).

**Proof.** It is equivalent to show that \( \mathcal{A}^{i-1} D_I(-) \approx (\mathcal{A}^{i-1} D_{I_0}(-))_y \). We can calculate \( \mathcal{A}^{i-1} D_I(M) \) as \( H^{i-1}(D_I(E^\bullet)) \) where \( M \to E^\bullet \) is an injective resolution, but by **Proposition 2.1**, \( D_I(-) \approx D_{I_0}(-)_y \) where \( (-)_y \) commutes with the formation of cohomology. Thus,

\[ H^{i-1}(D_I(E^\bullet)) \approx H^{i-1}(D_{I_0}(E^\bullet))_y = (\mathcal{A}^{i-1} D_{I_0}(M))_y. \]

\( \square \)

**Theorem 2.1.** Let \( R \) be a locally almost factorial Noetherian normal ring, and \( I, J \) be ideals of \( R \). The set \( \text{Ass} H^2_I(J) \) is finite.

**Proof.** \( R \) is a product of normal domains \( R_1 \times \cdots \times R_k \), and \( J \) is a product \( J_1 \times \cdots \times J_k \) with \( J_i \subseteq R_i \). It is enough to show that \( \text{Ass} H^2_{I_0}(J_i) \) is finite for all \( i \), so assume that \( R \) is a domain. By **Lemma 2.1**, we need only deal with the case in which \( \text{depth}_I(R) = 1 \). It is enough to show that \( \text{Ass} H^2_{I_0}(J_f) \) is finite for each chart \( \text{Spec}(R_f) \) in a finite cover of \( \text{Spec}(R) \). By **Corollary 2.1**, working with one chart at a time, and replacing \( R \) by \( R_f \) and \( I \) by an ideal with the same radical, we may assume that \( I \) has the form \( I = yR \cap I_0 \) where \( \text{depth}_{I_0}(R) \geq 2 \). By **Corollary
2.2, this decomposition gives \( H^2_J(J) \simeq H^2_{I_0}(J)_y \). It is therefore enough to show that \( H^2_{I_0}(J) \) is finite. But \( \text{depth}_{I_0}(R) \geq 2 \), so this follows at once from Lemma 2.1.

3. An example of an infinite \( \text{Ass} H^3_I(J) \)

Let \( K \) be a field, \( A = K[[s, t, u, v, x, y]] \), and \( f = sv^2 x^2 - (s + t)vxy + tu^2 y^2 \). Katzman [Kat02] showed that \( \text{Ass} H^2_{(u,v)}(A/f) \) is infinite. The hypersurface \( S = A/f \) can be presented as a complete intersection of higher codimension. For example, if \( R = A[[z]] \) and \( J = (z, f)R \), then clearly \( S \simeq R/J \). Also, if \( I = (z, f, u, v)R \), then \( IS = (u, v)S \) and thus \( H^1_I(S) \simeq H^i_{(u,v)}(S) \) for all \( i \). In this case, \( \text{depth}_I(R) = 3 \) (the sequence \( z, f, u \in I \) is \( R \)-regular), so the long exact sequence from applying \( \Gamma_I(-) \) to \( 0 \to J \to R \to S \to 0 \) begins with

\[
\begin{array}{cccccc}
0 & \to & H^1_J & \to & H^1_{(u,v)}(S) & \to 0 \\
& & H^2_J & \to & H^2_{(u,v)}(S) & \\
& & H^3_J & \to & H^3_I(R) & \to 0 \\
& & H^4_J & \to & H^4_I(R) & \to 0 \\
& & \vdots & \vdots & \vdots & \\
\end{array}
\]

From this, we see that \( H^2_{(u,v)}(S) \) is isomorphic to a submodule of \( H^3_I(J) \), where

\[
S \simeq \frac{K[[s, t, u, v, x, y]]}{sv^2 x^2 - (s + t)vxy + tu^2 y^2}.
\]

\( \text{Ass} H^2_{(u,v)}(S) \) is infinite, so \( \text{Ass} H^3_I(J) \) must be infinite as well. Note that this example occurs when \( J \) is generated by a regular sequence of length 2.

This approach can be used to produces examples of parameter ideals \( J \) with a similar relationship to \( R \) and \( S \). If \( S = A/J_0 \) is a complete intersection of dimension \( n \), with \( A \) an LC-finite regular ring and \( J_0 \) an ideal generated by a regular sequence, we can let \( R = A[z_1, \ldots, z_N] \) for \( N \gg 0 \), and let \( J = (z_1, \ldots, z_N)R + J_0R \), and will have \( S \simeq R/J \). For any ideal \( I_0 \subseteq A \), set \( I = J + I_0R \) and \( H^i_{I_0}(S) \simeq H^i_I(S) \), with \( d = \text{depth}_I(R) > n + 1 \) if \( N \) is large enough. Using the long exact sequence from applying \( \Gamma_I(-) \) to \( 0 \to J \to R \to S \to 0 \), it follows at once that \( H^i_I(J) \simeq H^i_{I_0} - 1(S) \) for all \( 1 \leq i \leq n + 1 \), that \( H^i_I(J) = 0 \) for \( n + 2 \leq i \leq d - 1 \), and that \( H^i_I(J) \simeq H^i_I(R) \) for all \( i \geq d \). In sufficiently large cohomological degrees, \( \text{Ass} H^i_I(J) \) is finite, and in degrees \( 1 \leq i \leq n + 1 \), the local cohomology of \( J \) has exactly the same finiteness properties as the local cohomology of \( S \).

4. Finiteness of \( \text{Ass} H^1_I(J) \) vs finiteness of \( \text{Ass} H^{i-1}_I(R/J) \)

The class of examples presented in Section 3 may be somewhat unsatisfying, since the infinite collection of embedded primes we found in \( H^1_I(J) \) was directly inherited from \( H^{i-1}_I(R/J) \). It is an interesting and unresolved question whether there exist modules \( H^1_I(J) \) that exhibit this type of behavior even when \( H^{i-1}_I(R/J) \) is well-controlled. To be precise, the following question is, to the best of our knowledge, still open:
Question 3. Let $R$ be an LC-finite regular ring, $J \subseteq R$ be an ideal generated by a regular sequence, and $I$ be an ideal of $R$ containing $J$. Does the finiteness of Ass $H^{-1}_I(R/J)$ imply the finiteness of Ass $H^i_I(J)$?

If $J$ is generated by a regular sequence of length 1, then $J \cong R$ as an $R$-module, and Ass $H^1_I(J)$ is finite, so the claim is trivially true. We therefore restrict our attention to when depth$_J(R) \geq 2$. We think of $i$ as being fixed with $I$ varying. The case $i = 2$ is completely answered, since Ass $H^2_I(R/J)$ is finite (as is true of $H^1_I(M)$ for any finitely generated module $M$), and Ass $H^2_I(J)$ is finite by Theorem 2.1. The goal of this section is to give a partial positive answer to this question when $i = 3$ and when $i = 4$. As $i$ gets larger, our results require increasingly restrictive hypotheses on $R/J$.

To begin, notice that we can very easily ignore ideals $I$ where the depth of $R$ on $I$ is too large.

Lemma 4.1. Let $R$ be a Noetherian ring, let $I$ and $J$ be ideals of $R$, and let $S = R/J$. Fix $i \geq 1$ and assume Ass $H_i^1(R)$ is finite. If depth$_I(R) > i$, then Ass $H_i^1(J)$ is finite if and only if Ass $H_i^{i-1}(R/J)$ is finite.

Proof. There is a short exact sequence

$$0 \rightarrow H_i^{i-1}(R/J) \rightarrow H_i^1(J) \rightarrow N \rightarrow 0$$

where $N \subseteq H_i^1(R)$, so Ass $N$ is finite. □

We may therefore restrict our focus to the case where depth$_I(R) \leq i - 1$. We will show that it actually suffices to only consider ideals $I$ such that depth$_I(R) = i - 1$. This crucial simplification is inspired by a very similar strategy employed in Hellus [Hel01]. Before proceeding, we require a notion of parameters in a global ring, and the following lemma provides one suitable enough for use in our main proofs.

Lemma 4.2. Let $R$ be a Noetherian ring, let $I$ be a proper ideal of height $h \geq 0$, and let $J \subseteq I$ be an ideal of height $j \geq 0$.

(a) If an ideal of the form $(x_1, \ldots, x_h)R$ has height $h$, then it has pure height $h$.
(b) Any sequence $x_1, \ldots, x_j \in J$ generating an ideal of height $j$ (including the empty sequence if $j = 0$) can be extended to a sequence $x_1, \ldots, x_h \in I$ generating an ideal of height $h$.
(c) There is a sequence $x_1, \ldots, x_h \in I$ such that $(x_1, \ldots, x_h)R$ has (necessarily pure) height $h$.

Proof. (a) Every minimal prime of a height $h$ ideal has height at least $h$, and every minimal prime of a $h$-generated ideal has height at most $h$.

(b) If $j = h$, there is nothing to do, so assume $j < h$. By induction, it is enough to show that we can extend the sequence by one element. Since $j < h$, $I$ is not contained in any minimal prime of $(x_1, \ldots, x_j)R$ (all of which have height $j$), and so we may choose $x \in I$ avoiding all such primes. A height $j$ prime containing $(x_1, \ldots, x_j)R$ therefore cannot also contain $xR$. Thus, the minimal primes of $(x, x_1, \ldots, x_j)R$ have height at least $j + 1$. By Krull’s height theorem, they also have height at most $j + 1$.

(c) This follows at once from (b) by taking $J = (0)$. □

By convention, we take the height of the unit ideal to be $+\infty$. An intersection of prime ideals of $R$ indexed by the empty set is taken to be $\bigcap_{i \in \emptyset} P_i = R$.

Lemma 4.3. Let $R$ be a Noetherian ring, let $I$ be a proper ideal of height $h$, and let $J \subseteq I$ be an ideal of height $j \leq h$ generated by $j$ elements. There is an element $y \in R$ that satisfies the following properties.
Theorem 4.1. Let $R$ be a Noetherian ring, $I$ be an ideal of height $h$, and $J \subseteq I$ be an ideal of height $j \geq 0$ generated by $j$ elements. For any $k \geq 0$, there is an ideal $I_{k,J} \supseteq I$ such that

- $\text{ht}(I_{k,J}) \geq \text{ht}(I) + k$
- The natural transformation $H^i_{I_{k,J}}(-) \rightarrow H^i_I(-)$

is an isomorphism on $R$-modules for all $i > h + k$, and an isomorphism on $R/J$-modules for all $i > h - j + k$. If $i = h + k$ (resp. $i = h - j + k$) this natural transformation is a surjection on $R$-modules (resp. $R/J$-modules).

Proof. If $k = 0$, choose $I_{0,J} = I$. Fix $k \geq 1$, and suppose that we’ve chosen the ideal $I_{k-1,J}$ by induction. We must choose $I_{k,J}$. For brevity, we will suppress $J$ from our notation, and write $I_{k-1}$ and $I_k$ for $I_{k-1,J}$ and $I_{k,J}$, respectively.

If $\text{ht}(I_{k-1}) > h + k - 1$ we can simply pick $I_k = I_{k-1}$, so assume that $\text{ht}(I_{k-1}) = h + k - 1$. By Lemma 4.3 there is an element $y \in R$ such that $\text{ht}(yR + I_{k-1}) \geq (h + k - 1) + 1$, with $\text{ara}_R(yR \cap I_{k-1}) \leq h + k - 1$ and $\text{ara}_{R/J}(y(R/J) \cap I_{k-1}/J) \leq (h + k - 1) - j$
Consider the Mayer-Vietoris sequence

\[
\begin{array}{cccc}
H^i_{yR+I_{k-1}}(-) & \rightarrow & H^i_{yR}(-) \oplus H^i_{I_{k-1}}(-) & \rightarrow \\
\cdots & & \cdots & \\
H^{i-1}_{yR\cap I_{k-1}}(-) & \rightarrow & H^{i-1}_{yR}(-) & \\
\end{array}
\]

Let \( i > h + k \). Since \( i - 1 > \text{ara}_R(yR \cap I_{k-1}) \), we get vanishing \( H^{i-1}_{yR\cap I_{k-1}}(-) = H^i_{yR\cap I_{k-1}}(-) = 0 \).

Since \( i \geq h + k + 1 \geq 2 \), we also have \( H^i_{yR}(-) = 0 \), and therefore obtain a natural isomorphism \( H^i_{yR+I_{k-1}}(-) \cong H^i_{I_{k-1}}(-) \). Notice that if \( i = h + k \), then we still have \( H^i_{yR\cap I_{k-1}}(-) = 0 \), so

\[
H^i_{yR+I_{k-1}}(-) \rightarrow H^i_{yR}(-) \oplus H^i_{I_{k-1}}(-) \rightarrow 0
\]

is exact, implying that the component map \( H^i_{yR+I_{k-1}}(-) \rightarrow H^i_{I_{k-1}}(-) \) is surjective. Working with \( R/J \)-modules, an identical argument using the fact that

\[
\text{ara}_{R/J}(yR \cap I_{k-1}/J) \leq (h + k - 1) - j
\]

shows that \( H^i_{y(R/J)+I_{k-1}/J}(-) \cong H^i_{I_{k-1}/J}(-) \) when \( i > h + k - j \) and \( H^i_{y(R/J)+I_{k-1}/J}(-) \rightarrow H^i_{I_{k-1}/J}(-) \) when \( i = h + k - j \). Finally, \( \text{ht}(yR + I_{k-1}) \geq h + k \), so we may in fact choose \( I_k = yR + I_{k-1} \), which completes the induction. \( \square \)

One immediate application of this theorem is a generalization of Corollary 2 in [Hel01]. This generalization provides a new proof of a result from Proposition 2.3 in Marley [Mar01], namely, for any Noetherian ring \( R \), any ideal \( I \subseteq R \), and any \( R \)-module \( M \), \( \{ P \in \text{Supp} H^i_I(M) \mid \text{ht}(P) = i \} \) is a finite set. Since our result comes from a surjection of functors, we will describe it in terms of the “support” of \( H^i_I(-) \).

By the support of a functor \( F : \text{Mod}_R \rightarrow \text{Mod}_R \), we mean the set of primes \( P \in \text{Spec}(R) \) such that \( F(-)_P \) is not the zero functor. That is to say,

\[
\text{Supp}(F) := \{ P \in \text{Spec}(R) \mid \exists M \in \text{Mod}_R \text{ such that } F(M)_P \neq 0 \}
\]

For example, if \( J \subseteq R \) is an ideal and \( i \geq 0 \), then \( \text{Supp} H^i_I(-) \subseteq V(I) \). If \( i > \text{ht}(I) \), this inclusion is not sharp. The following Corollary shows us how to find a strictly smaller closed set containing \( \text{Supp} H^i_I(-) \).

**Corollary 4.1.** Let \( R \) be a Noetherian ring and \( I \) be an ideal. Then for all \( i \geq 0 \), there is an ideal \( I' \supseteq I \) with \( \text{ht}(I') \geq i \) such that \( \text{Supp} H^i_I(-) \subseteq V(I') \). In particular, for any \( R \)-module \( M \), the set

\[
\text{Supp} H^i_I(M) \cap \{ P \in \text{Spec}(R) \mid \text{ht}(P) = i \}
\]

is a subset of \( \text{Min}_R(R/I') \), and is therefore finite. If \( R \) is semilocal and \( i \geq \dim(R) - 1 \), then \( \text{Supp} H^i_I(M) \) is a finite set.

**Proof.** Fix \( i \geq 0 \) and write \( h = \text{ht}(I) \). If \( i < h \), then because \( \text{Supp} H^i_I(-) \subseteq V(I) \), we already have \( \text{ht}(P) \geq h > i \) for all \( P \in \text{Supp} H^i_I(-) \) and there is nothing to prove. So assume that \( i \geq h \). By Theorem 4.1 (in the case where \( k = i - h \)), there is an ideal \( I' \supseteq I \) such that \( \text{ht}(I') \geq i \) and \( H^i_{I'}(-) \rightarrow H^i_I(-) \). In particular, for any \( R \)-module \( M \), \( H^i_I(M) \) is \( I' \)-torsion, and thus \( \text{Supp} H^i_I(M) \subseteq V(I') \). All primes in \( V(I') \) have height at least \( i \). Any primes of height exactly \( i \) must be among the minimal primes of \( I' \), of which there are only finitely many. \( \square \)

Of primary interest within the scope of this paper is the following application of Theorem 4.1 to the setting of Question 3.
Corollary 4.2. Let $R$ be a Cohen-Macaulay ring, and let $J \subseteq R$ be an ideal generated by a regular sequence of length $j \geq 1$. Fix a nonnegative integer $i$, and let $I$ be an ideal containing $J$ such that $\text{depth}_I(R) \leq i - 1$. Then there is an ideal $I' \supseteq I$ such that $\text{depth}_{I'}(R) \geq i - 1$ and such that $H^j_{I'}(R) \simeq H^j_I(J)$ and $H^{j-1}_{I'}(R/J) \simeq H^{j-1}_I(R/J)$.

Proof. Write $h = \text{ht}(I)$ and $j = \text{ht}(J)$. Apply Theorem 4.1 with $k = i - 1 - h$ to obtain $I' \supseteq I$ such that $\text{depth}_{I'}(R) = \text{ht}(I') \geq i - 1$ and such that the natural transformation $H^j_{I'}(-) \to H^j_I(-)$ is an isomorphism on $R$-modules whenever $\ell > i - 1$ and on $R/J$-modules whenever $\ell > i - 1 - j$. In particular, we see that $H^j_{I'}(-) \to H^j_I(-)$ is an isomorphism on $R$-modules $H^{j-1}_I(-) \to H^{j-1}_I(-)$ is an isomorphism on $R/J$-modules.

In light of this corollary, what are the main cases of interest for Question 3? Assume that $R$ is Cohen-Macaulay, $J \subseteq R$ is generated by a regular sequence of length $j \geq 2$, and $I \supseteq J$ is any ideal. Fix $i \geq 0$ and let $a = \text{depth}_I(R/J) = \text{depth}_I(R) - j$. If $a + j \leq i - 1$, then by Corollary 4.2, we can replace $I$ with a possibly larger ideal in order to assume that $a + j \geq i - 1$, without affecting $H^j_I(J)$ and $H^{j-1}_I(R/J)$. Lemma 4.1 gives a positive answer to Question 3 if $a + j > i - 1$, so we may assume that $a + j = i - 1$. Note in particular that this allows us to ignore all values of $i$ and $j$ for which $j > i - 1$. Here is a table illustrating the relevant values of $a$ to consider for various small values of $i$ and $j$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$a$</th>
<th>$j = 2$</th>
<th>$a = 0$</th>
<th>$a = 1$</th>
<th>$a = 2$</th>
<th>$a = 3$</th>
<th>$a = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 3$</td>
<td>$j = 3$</td>
<td>$\emptyset$</td>
<td>$a = 0$</td>
<td>$a = 1$</td>
<td>$a = 2$</td>
<td>$a = 3$</td>
<td>$a = 2$</td>
<td>$a = 1$</td>
</tr>
<tr>
<td>$i = 4$</td>
<td>$j = 4$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$a = 0$</td>
<td>$a = 1$</td>
<td>$a = 2$</td>
<td>$a = 3$</td>
<td>$a = 2$</td>
</tr>
<tr>
<td>$i = 5$</td>
<td>$j = 5$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$a = 0$</td>
<td>$a = 1$</td>
<td>$a = 2$</td>
<td>$a = 3$</td>
</tr>
<tr>
<td>$i = 6$</td>
<td>$j = 6$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$a = 0$</td>
<td>$a = 1$</td>
<td>$a = 2$</td>
</tr>
<tr>
<td>$i = 7$</td>
<td>$j = 7$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$a = 0$</td>
<td>$a = 1$</td>
</tr>
</tbody>
</table>

We will attack the cases $a = 0$ and $a = 1$, and the next lemma is our main tool in doing so.

Lemma 4.4. Fix $a \geq 0$. Let $R$ be a Noetherian ring, let $J \subseteq R$ be an ideal generated by a regular sequence of length $j \geq 2 - a$, let $I$ be an ideal containing $J$, and let $S = R/J$. Suppose that $IS$ can be decomposed (up to radicals) as $yS \cap I_0$ with $\text{depth}_{I_0}(S) > a$. Suppose further that $\text{Ass} H^{j+a+1}_I(S)$ is finite. Then $\text{Ass} H^{j+a+1}_I(J)$ is finite if and only if $\text{Ass} H^{j+a+1}_I(S)$ is finite.

Proof. By Corollary 2.2, there is a natural isomorphism $H^i_{I_0}(S_y) \simeq H^i_I(S)$ for all $i \geq 2$, so that in particular, $H^{j+a}_I(S)$ is an $S_y$-module. The natural map $\psi : H^{j+a}_I(S) \to H^{j+a}_I(S)$ therefore factors through $H^{j+a}_I(R) \to S_y \otimes_R H^{j+a}_I(R)$ to give an $S_y$-linear map $S_y \otimes_R H^{j+a}_I(R) \to H^{j+a}_I(S)$.

We claim that $\psi = 0$, and for this it suffices to show that $S_y \otimes_R H^{j+a}_I(R) = 0$.

Consider the decomposition of $I$ up to radicals as $yS \cap I_0$ in $S$. We can replace $y$ by some lift mod $J$ to assume that $y \in R$, and since $I_0$ is expanded from $R$, we can write $I_0 = I_0' + J$, where $I_0'$ is an ideal containing $J$. We therefore have $I = (yR + J) \cap I_0$ in $R$ (after possibly replacing $I$ by an ideal with the same radical). Note that $\text{depth}_{I_0}(R) > j + a$. We can
write $S_y \otimes_R H^{j+a}_I(R) = S_y \otimes_{R_y} H^{j+a}_{IR_y}(R_y)$ where $IR_y = (y + J)R_y \cap I_0^t R_y = I_0^t R_y$, and thus $H^{j+a}_{IR_y}(R_y) = H^{j+a}_{t_0}(R_y)$. Since $\text{depth}_{t_0}(R) > j + a$, we have $H^{j+a}_{t_0}(R) = 0$ and consequently, $\psi = 0$.

There is an exact sequence

$$0 \to H^{j+a}_I(S) \to H^{j+a+1}_I(J) \to H^{j+a+1}_I(R).$$

Since $H^{j+a+1}_I(R)$ is finite, the claim follows at once. $\square$

We can now prove the main result of this section.

**Theorem 4.2.** Let $R$ be an LC-finite regular ring, let $J \subseteq R$ be an ideal generated generated by a regular sequence of length $j \geq 2$, and let $S = R/J$. For any $I$ of $R$ such that $I \supseteq J$,

(i) $\text{Ass } H^i_I(J)$ and $\text{Ass } H^{i-1}_I(S)$ are always finite for $i \leq 2$.

(ii) If the irreducible components of $\text{Spec}(S)$ are disjoint, then $\text{Ass } H^3_I(J)$ is finite if and only if $\text{Ass } H^2_I(S)$ is finite.

(iii) If $S$ is normal and locally almost factorial, then $\text{Ass } H^1_I(J)$ is finite if and only if $\text{Ass } H^2_I(S)$ is finite.

**Proof.** For (i), it is well known that for any finitely generated $R$-module $M$, $\text{Ass } H^i_I(M)$ is finite whenever $i \leq 1$. The finiteness of $\text{Ass } H^2_I(J)$ is the subject of Theorem 2.1.

For (ii), we may use Corollary 4.2 to replace $I$ with a possibly larger ideal in order to assume that $\text{depth}_I(R) \geq 2$. By Lemma 4.1, (ii) is immediate if $\text{depth}_I(R) > 2$, so assume $\text{depth}_I(R) = 2$. Since $J \subseteq I$ and $\text{depth}_J(R) \geq 2$, it follows that $\text{depth}_J(R) = 2$ and $\text{depth}_I(S) = \text{ht}(IS) = 0$. Let $e_1, \ldots, e_t \in S$ be a complete set of orthogonal idempotents. The minimal primes of $S$ are $\sqrt{(1 - e_1)S}, \ldots, \sqrt{(1 - e_t)S}$, and thus, every pure height 0 ideal of $S$ has arithmetic rank at most 1. Up to radicals, we can therefore write $IS$ as $yS \cap I_0$ where $\text{ht}(I_0) = \text{depth}_{t_0}(S) > 0$. Since $\text{depth}_J(R) \geq 2 - 0$, the claim follows from Lemma 4.4 in the case where $a = 0$.

For (iii), again using Corollary 4.2 and Lemma 4.1, we may assume that $\text{depth}_I(R) = \text{depth}_I(S) + \text{depth}_J(R) = 3$. Since $\text{depth}_J(R) \geq 2$, this means $\text{depth}_I(S) \leq 1$. If $\text{depth}_J(R) = 3$, giving $\text{depth}_I(S) = 0$, then we may argue as in the proof of (ii) (note that $S$ is a product of domains). If $\text{depth}_J(R) = 2$, giving $\text{depth}_I(S) = 1$, then by Corollary 2.1 there is a finite cover of $\text{Spec}(S)$ by charts $\text{Spec}(S_{f_1}), \ldots, \text{Spec}(S_{f_t})$ such that for each $i$, we can write (up to radicals) $IS_{f_i} = y_i S_i \cap I_{0,i}$ with $\text{depth}_{t_0,i}(S_{f_i}) > 1$. Replace $f_1, \ldots, f_t$ with lifts from $R/J$ to $R$ in order to assume $f_1, \ldots, f_t \in R$. Lemma 4.4 in the case $a = 1$ shows that for each $i$, $\text{Ass } H^1_I(J)_{f_i}$ is finite if and only if $\text{Ass } H^3_I(S)_{f_i}$ is finite. The charts $\text{Spec}(R_{f_1}), \ldots, \text{Spec}(R_{f_t})$ do not necessarily cover $\text{Spec}(R)$, but they do cover the subset $V(J)$. Since $I \supseteq J$, $\text{Supp } H^2_I(-) \subseteq V(I) \subseteq V(J)$ for all $\ell$, so showing that $\text{Ass } H^1_I(J)$ is finite is equivalent to showing that $\text{Ass } H^3_I(J)_{f_i}$ is finite for each $i$. The result we proved on each chart therefore implies $\text{Ass } H^1_I(J)$ is finite if and only if $\text{Ass } H^3_I(S)$ is finite. $\square$

Under the hypotheses of (iii), we can give the following partial answer to Question 3 for local rings of sufficiently small dimension.

**Corollary 4.3.** Let $(R, m, K)$ be an LC-finite regular local ring of dimension at most 7, let $J$ be an ideal generated by a regular sequence of length at least 2, and assume that $S = R/J$ is normal and almost factorial. Let $I$ be any ideal of $R$ containing $J$. Then for all $i \geq 1$, $\text{Ass } H^i_I(J)$ is finite if and only if $\text{Ass } H^{i-1}_I(S)$ is finite.
Proof. The case \( i \leq 4 \) is the subject of Theorem 4.2. We must have \( \dim(S) \leq 5 \) since \( \text{depth}_I(R) \geq 2 \), so by Corollary 4.1, \( \text{Supp} \, H^i_I(S) \) (and hence \( \text{Ass} \, H^i_I(S) \)) is a finite set if \( i - 1 \geq 4 \). Likewise, for any homomorphic image \( H^i_I(S) \to N \), the set \( \text{Supp}(N) \) is finite. There is an exact sequence \( 0 \to N \to H^i_I(J) \to M \to 0 \) where \( N \) is a homomorphic image of \( H^i_I(S) \) and \( M \) is a submodule of \( H^i_I(R) \). If \( i \geq 5 \), both \( \text{Ass}(N) \) (a subset of \( \text{Supp}(N) \)) and \( \text{Ass}(M) \) (a subset of \( \text{Ass} \, H^i_I(R) \)) are finite, so \( \text{Ass} \, H^i_I(J) \) is finite as well. \( \Box \)

5. Regular Parameter Ideals in Characteristic \( p > 0 \)

In this section, we will show that if \( R \) is a regular ring of prime characteristic \( p > 0 \) and \( J \) is an ideal such that \( R/J \) is regular, then for any ideal \( I \subseteq R \) and any \( i \geq 0 \), the set \( \text{Ass} \, H^i_I(J) \) is finite. This result is essentially a corollary of a stronger result. Specifically, we will show that if \( R \to S \) is a homomorphism between two regular rings of prime characteristic \( p > 0 \), and \( I \subseteq R \) is an ideal, then for any \( i \geq 0 \), the natural map

\[
S \otimes_R H^i_I(R) \to H^i_I(S)
\]

is a morphism of \( F \)-finite \( \mathcal{F}_S \)-modules in the sense of Lyubeznik [Lyu97]. The proof of this statement relies on understanding a certain family of natural transformations that compare the cohomology of a complex before and after applying some base change functor.

5.1. Natural transformations associated with cohomology and base change. Let \( \text{Mod}_R \) denote the category of modules over a ring \( R \), and let \( \text{Kom}_R \) denote the category of cohomologically indexed complexes of \( R \)-modules. Let \( H^i : \text{Kom}_R \to \text{Mod}_R \) denote the functor that takes a complex \( C^* \) to its \( i \)th cohomology module \( H^i(C^*) \).

**Definition 5.1.** If \( f : R \to S \) is a ring homomorphism and \( C^* \) is an \( R \)-complex, then the natural (\( R \)-linear) map \( C^* \to S \otimes_R C^* \) induces \( H^i(\cdot) \to H^i(S \otimes_R C^*) \), which factors uniquely through the natural map \( H^i(\cdot) \to S \otimes_R H^i(\cdot) \) to an \( S \)-linear map \( S \otimes_R H^i(\cdot) \to H^i(S \otimes_R C^*) \). Call this map \( h^i_f(\cdot) \), and let \( h^i_f \) denote the corresponding natural transformation

\[
h^i_f(\cdot) : S \otimes_R H^i(\cdot) \to H^i(S \otimes_R \cdot)
\]

of functors \( \text{Kom}_R \to \text{Mod}_S \).

If the homomorphism \( f : R \to S \) is understood from context, we will write \( h^i_{S/R}(\cdot) \) instead of \( h^i_f(\cdot) \). The latter, more precise notation is reserved for ambiguous cases, such as when \( R = S \) has prime characteristic \( p \) and \( f \) is the Frobenius homomorphism.

Note that \( S \) is flat over \( R \) if and only if \( h^i_{S/R}(\cdot) \) is an isomorphism of functors. Our goal in this section is to briefly review some straightforward but crucially important compatibility properties of these natural transformations.

**Proposition 5.1.** [Compatibility with composition] Let \( R \to S \to T \) be ring homomorphisms. The following diagram of functors \( \text{Kom}_R \to \text{Mod}_T \) commutes

\[
T \otimes_R H^i(\cdot) \\
\downarrow h^i_{T/R}(\cdot) \\
T \otimes_S (S \otimes_R H^i(\cdot)) \\
\downarrow \text{id}_T \otimes_S h^i_{S/R}(\cdot) \\
T \otimes_S H^i(S \otimes_R \cdot) \\
\downarrow h^i_{T/S}(S \otimes_R \cdot) \\
H^i(T \otimes_R \cdot)
\]

\[
\begin{array}{c}
H^i(T \otimes_R \cdot) \\
\end{array}
\]
where the equalities shown above come from identifying the functor $T \otimes_S (S \otimes_R -)$ with $T \otimes_R -$.

**Proof.** The diagram for a single $R$-complex $C^\bullet$ is obtained by applying $H^i(-)$ to

$$
\begin{array}{ccc}
C^\bullet & \rightarrow & S \otimes_R C^\bullet \\
\downarrow & & \downarrow \\
T \otimes_S (S \otimes_R C^\bullet) & \cong & T \otimes_R C^\bullet
\end{array}
$$

and base changing the modules $H^i(C^\bullet)$ and $H^i(S \otimes_R C^\bullet)$ from $\text{Mod}_R$ and $\text{Mod}_S$ to $\text{Mod}_T$, respectively. Naturality of the resulting diagram is straightforward to verify. \qed

**Proposition 5.2.** [Simultaneous flat base change] Suppose there is a commutative square of ring homomorphisms

$$
\begin{array}{ccc}
R & \rightarrow & S \\
\downarrow & & \downarrow \\
R' & \rightarrow & S'
\end{array}
$$

such that $R'$ is flat over $R$, and $S'$ is flat over $S$. Then there is a commutative square of functors $\text{Kom}_R \rightarrow \text{Mod}_{S'}$

$$
\begin{array}{ccc}
S' \otimes_S (S \otimes_R H^i(-)) & \overset{id_{S'} \otimes_S h^i_{S/R}(-)}{\rightarrow} & S' \otimes_S H^i(S \otimes_R -) \\
\downarrow & & \downarrow \\
S' \otimes_{R'} H^i(R' \otimes_R -) & \overset{h^i_{S'/R'}(R' \otimes_R -)}{\rightarrow} & H^i(S' \otimes_{R'} (R' \otimes_R -))
\end{array}
$$

where the vertical arrows are natural isomorphisms.

**Proof.** Apply Proposition 5.1 to $R \rightarrow S \rightarrow S'$ in order to see that the upper right corner of the below diagram commutes, and then to $R \rightarrow R' \rightarrow S'$ to see that the lower left corner commutes as well:
We are finished upon observing that the flatness of $R'$ over $R$ (resp. of $S'$ over $S$) implies that $h^i_{R'/R}(\cdot)$ (resp. $h^i_{S'/S}(\cdot)$) is a natural isomorphism.

Our main interest is in applying Proposition 5.2 in the case where $R$ and $S$ are regular rings of primes characteristic $p > 0$, and $R \to R'$, $S \to S'$ are the ($e$-fold iterates of the) Frobenius homomorphisms of $R$ and $S$. Some notation: if $A$ is a ring of prime characteristic $p > 0$, let $F_A : A \to A$ denote the Frobenius homomorphism on $A$, and $\mathcal{F}_A(\cdot)$ (either $\text{Mod}_A \to \text{Mod}_A$ or $\text{Kom}_A \to \text{Kom}_A$, depending on context) denote the base change functor associated with $F_A$. Let $\mathcal{F}_A(\cdot)$ denote the $e$-fold iterate of $\mathcal{F}_A(\cdot)$.

**Corollary 5.1.** Let $R \to S$ be a homomorphism between two regular rings of prime characteristic $p > 0$. For each $q = p^e$, the following diagram commutes,

$$
\begin{array}{ccc}
S^i \otimes_S (S \otimes_R H^i(\cdot)) & \xrightarrow{id_S \otimes_S h^i_{S/R}(\cdot)} & S^i \otimes_S H^i(S \otimes_R \cdot) \\
\downarrow & & \downarrow h^i_{S/R}(S \otimes_R \cdot) \\
S^i \otimes_R H^i(\cdot) & \xrightarrow{h^i_{S/R}(\cdot)} & H^i(S^i \otimes_R \cdot) \\
\downarrow & & \downarrow \\
S^i \otimes_{R'} (R' \otimes_R H^i(\cdot)) & \xrightarrow{h^i_{S'/R'}(\cdot \otimes_{R'} \cdot)} & H^i(S^i \otimes_{R'} (R' \otimes_R \cdot)) \\
\end{array}
$$

where the vertical arrows are isomorphisms.

In other words, if we identify $\mathcal{F}_S^e(S \otimes_R H^i(\cdot))$ with $S \otimes_R H^i(\mathcal{F}_R^e(\cdot))$ and $\mathcal{F}_S^e(H^i(S \otimes_R \cdot))$ with $H^i(S \otimes \mathcal{F}_R^e(\cdot))$ using the vertical isomorphisms, then for all $e \geq 0$, $h^i_{S/R}(\mathcal{F}_R^e(\cdot)) = \mathcal{F}_S^e(h^i_{S/R}(\cdot))$. Concretely, if $C^\bullet$ is a complex of $R$-modules and $C^\bullet_e := \mathcal{F}_R^e(C^\bullet)$, then as long as $R$ and $S$ are both regular, the family $\{h^i_{S/R}(C^\bullet_e)\}_{e=0}^\infty$ of homomorphisms

$$S \otimes_R H^i(C^\bullet_e) \to H^i(S \otimes_R C^\bullet_e)$$

is determined by the homomorphism at $e = 0$ by taking Frobenius powers. For example, if $C^\bullet = K^\bullet(\underline{f}; R)$ is the Koszul complex on some sequence of elements $\underline{f} = f_1, \cdots, f_t \in R$, and if $\underline{f}^{[0]} := f_1^0, \cdots, f_t^0$, then the aforementioned family consists of the natural maps

$$h^i_e : S \otimes_R H^i(\underline{f}^{[0]}; R) \to H^i(\underline{f}^{[0]}; S).$$
At the level of Koszul cohomology, the claim that the entire family of maps \( \{ h^i_\epsilon \}_{\epsilon = 0}^\infty \) is determined by the data of just the 0th map \( h_0^i \) makes the utility of Corollary 5.1 intuitively clear. The statement is more abstruse when applied directly to the Čech complex, since \( \hat{C}(\mathbf{f}; R) \) is canonically identified with \( \mathcal{F}_R(\hat{C}(\mathbf{f}; R)) \). However, once we replace the role of \( R \) with a general \( F \)-finite \( F_R \)-module \( \mathcal{M} \) in the sequel, it will nonetheless simplify our main proofs if we work at the level of Čech cohomology.

5.2. **The natural map on local cohomology.** Throughout this section, \( R \) and \( S \) are regular rings of prime characteristic \( p > 0 \). Recall that an \( F_R \)-module is a pair \( (\mathcal{M}, \theta) \) consisting of an \( R \)-module \( \mathcal{M} \) and an isomorphism \( \theta : \mathcal{M} \to \mathcal{F}_R(\mathcal{M}) \). A morphism of \( F_R \)-modules \( (\mathcal{M}, \theta) \to (\mathcal{N}, \phi) \) is an \( R \)-linear map \( h : \mathcal{M} \to \mathcal{N} \) such that \( \phi \circ h = \mathcal{F}_R(h) \circ \theta \). If \( R \to S \) is a ring homomorphism and \( (\mathcal{M}, \theta) \) is an \( F_R \)-module, then \( S \otimes_R \mathcal{M} \) is an \( F_S \)-module via the structure isomorphism \( \text{id}_S \otimes \theta : S \otimes_R \mathcal{M} \to S \otimes_R \mathcal{F}_R(\mathcal{M}) \), where we identify the \( S \otimes_R \mathcal{F}_R(\mathcal{M}) \) with \( \mathcal{F}_S(S \otimes_R \mathcal{M}) \) in the canonical way.

Let \( \mathbf{f} = f_1, \cdot \cdot \cdot , f_i \) be a sequence of elements of \( R \), let \( C^* := \hat{C}^*(\mathbf{f}; R) \) be the Čech complex on \( R \) associated with \( \mathbf{f} \). For shorthand, if \( M \) is an \( R \)-module, denote by \( C^*_{\mathcal{M}} := C^* \otimes_R M = \hat{C}^*(\mathbf{f}; \mathcal{M}) \) the Čech complex on \( M \) associated with \( \mathbf{f} \). The complex \( \mathcal{F}_R(C^*) \) is canonically identified with \( C^* \) itself, and likewise, for any \( R \)-module \( \mathcal{M} \), \( \mathcal{F}_R(C^*_{\mathcal{M}}) \) can be canonically identified with \( C^*_{\mathcal{F}_R(\mathcal{M})} \). If \( (\mathcal{M}, \theta) \) is an \( F_R \)-module, then applying \( C^* \otimes_R - \to \theta : \mathcal{M} \to \mathcal{F}_R(\mathcal{M}) \) gives an isomorphism of complexes \( \Theta : C^*_{\mathcal{M}} \to \mathcal{F}_R(C^*_{\mathcal{M}}) \). Using this isomorphism, \( H^i_1(\mathcal{M}) \) can naturally be made into an \( F_R \)-module for any \( i \): it’s structure isomorphism \( \Psi : H^i_1(\mathcal{M}) \to \mathcal{F}_R(H^i_1(\mathcal{M})) \) is given by the composition

\[
(5.1) \quad H^i_1(C^*_\mathcal{M}) \xrightarrow{H^i(\Theta)} H^i(\mathcal{F}_R(C^*_\mathcal{M})) \xrightarrow{h^i_{\mathcal{F}_R(C^*_\mathcal{M})}} \mathcal{F}_R(H^i(C^*_\mathcal{M}))
\]

The reader is referred to [Lyu97] for more about the induced \( F \)-module structure on local cohomology. A fundamentally important property is that if \( (\mathcal{M}, \theta) \) is \( F \)-finite, then \( (H^i_1(\mathcal{M}), \Psi) \) is \( F \)-finite.

We now prove our main compatibility result.

**Theorem 5.1.** Let \( R \) and \( S \) be regular rings of prime characteristic \( p > 0 \), fix an ideal \( I \subseteq R \) and an index \( i \geq 0 \), and let \( (\mathcal{M}, \theta) \) be an \( F_R \)-module. Then the natural map

\[
S \otimes_R H^i_1(\mathcal{M}) \to H^i_1(S \otimes_R \mathcal{M})
\]

is a morphism of \( F_S \)-modules.

**Proof.** Let \( C^*_{\mathcal{M}} = \hat{C}^*(\mathbf{f}; \mathcal{M}) \) be the Čech complex on \( \mathcal{M} \) associated with a sequence of elements \( \mathbf{f} = f_1, \cdot \cdot \cdot , f_i \) generating \( I \). It is enough to show that the diagram

\[
\begin{array}{ccc}
\mathcal{F}_S(S \otimes_R H^i_1(C^*_\mathcal{M})) & \xrightarrow{\mathcal{F}_S(h^i_{S/R}(C^*_\mathcal{M}))} & \mathcal{F}_S(H^i_1(S \otimes_R C^*_\mathcal{M})) \\
\downarrow i & & \downarrow i \\
S \otimes_R H^i_1(C^*_\mathcal{M}) & \xrightarrow{h^i_{S/R}(C^*_\mathcal{M})} & H^i_1(S \otimes_R C^*_\mathcal{M})
\end{array}
\]

commutes, where the vertical arrows are the inverses of the structure morphisms of \( S \otimes_R H^i_1(\mathcal{M}) \) and \( H^i_1(S \otimes_R \mathcal{M}) \) as \( F_S \)-modules, respectively. Let \( \Theta : C^*_\mathcal{M} \to \mathcal{F}_R(C^*_\mathcal{M}) \) denote the isomorphism of complexes induced by \( \theta \). Using the decomposition of the structure isomorphism of \( H^i_1(\mathcal{M}) \) in diagram (5.1), the stated result is equivalent to showing that the following diagram commutes.
Theorem 5.2. Let \( \text{Corollary 5.1} \) applied to the complex \( C_M \). The square of maps in the bottom two rows is induced from the diagram that results from applying \( H^i(-) \) to

\[
\begin{array}{ccc}
\mathcal{F}_R(C_M^\bullet) & \xrightarrow{\text{nat}} & S \otimes_R \mathcal{F}_R(C_M^\bullet) \\
\Theta^{-1} \downarrow & & \downarrow (\text{id}_S \otimes \Theta)^{-1} \\
C_M^\bullet & \xrightarrow{\text{nat}} & S \otimes_R C_M^\bullet
\end{array}
\]

Recall that \( C_M^\bullet = C^\bullet \otimes_R M \) and \( \mathcal{F}_R(C_M^\bullet) = C^\bullet \otimes_R \mathcal{F}_R(M) \), where \( C^\bullet = C^\bullet(f; R) \), so that the above diagram is \( C^\bullet \otimes_R - \) applied to

\[
\begin{array}{ccc}
\mathcal{F}_R(M) & \xrightarrow{\text{nat}} & S \otimes_R \mathcal{F}_R(M) \\
\Theta^{-1} \downarrow & & \downarrow (\text{id}_S \otimes \Theta)^{-1} \\
M & \xrightarrow{\text{nat}} & S \otimes_R M
\end{array}
\]

This final diagram obviously commutes.

\[ \Box \]

Corollary 5.2. Let \( R \) be a regular ring of prime characteristic \( p > 0 \), let \( J \subseteq R \) be an ideal such that \( R/J \) is regular, and let \( M \) be an \( F_R \)-module. For any ideal \( I \subseteq R \) and any \( i \geq 0 \), the natural map \( H^i_I(M)/JH^i_I(M) \to H^i_I(M/JM) \) is an \( F_{R/J} \)-module morphism.

Theorem 5.2. Let \( R \) be a regular ring of prime characteristic \( p > 0 \), let \( J \subseteq R \) be an ideal such that \( R/J \) is regular, and let \( M \) be an \( F \)-finite \( F_R \)-module (e.g. \( M = R \)). For any ideal \( I \subseteq R \) and any \( i \geq 0 \),

\[
\text{Coker} \left( H^i_I(M) \to H^i_I(M/JM) \right)
\]

is an \( F \)-finite \( F_{R/J} \)-module, and hence, has finitely many associated primes. The module \( H^i_I(JM) \) also has finitely many associated primes.

Proof. The map \( H^i_I(M) \to H^i_I(M/JM) \) factors through \( H^i_I(M) \to H^i_I(M)/JH^i_I(M) \), and since \( R \to R/J \) is surjective, the images of \( H^i_I(M) \) and \( H^i_I(M)/JH^i_I(M) \) inside \( H^i_I(M/JM) \) are equal. So,

\[
\text{Coker} \left( H^i_I(M) \to H^i_I(M/JM) \right) = \text{Coker} \left( H^i_I(M)/JH^i_I(M) \to H^i_I(M/JM) \right)
\]

where the latter is an \( F \)-finite \( F_{R/J} \)-module by Corollary 5.2 and Theorem 2.8 of [Lyu97]. The claim about associated primes of the cokernel follows from Theorem 2.12(a) of [Lyu97].
Regarding the claim about the associated primes of $H^i_I(J\mathcal{M})$, by applying $\Gamma_I(-)$ to $0 \to JM \to \mathcal{M} \to \mathcal{M}/JM \to 0$, the following sequence is exact.

$$\cdots \to H^{i-1}_I(\mathcal{M}) \to H^{i-1}_I(\mathcal{M}/JM) \to H^i_I(\mathcal{M}) \to H^i_I(J\mathcal{M}) \to \cdots$$

We therefore have a short exact sequence

$$0 \to \text{Coker} \left( H^{i-1}_I(\mathcal{M}) \to H^{i-1}_I(\mathcal{M}/JM) \right) \to H^i_I(J\mathcal{M}) \to N \to 0$$

for some submodule $N \subseteq H^i_I(\mathcal{M})$. Since $\text{Ass} H^i_I(\mathcal{M})$ is finite, so is $\text{Ass} N$, and the stated result now follows at once from the finiteness of $\text{Ass} \text{Coker} \left( H^{i-1}_I(\mathcal{M}) \to H^{i-1}_I(\mathcal{M}/JM) \right)$. □

Let $(R, m, K)$ be a regular local ring. Recall that a parameter ideal $J \subseteq R$ is called regular if it is generated by an $R$-regular sequence whose images in $m/m^2$ are linearly independent over $K$. Every ideal $J$ such that $R/J$ is regular has this form. If $R$ is complete and contains a field, then by then Cohen Structure Theorem, all examples of regular parameter ideals are isomorphic to an example of the form $R = K[[x_1, \cdots, x_m, z_1, \cdots, z_n]]$ and $J = (x_1, \cdots, x_m)R$ for some $m, n \geq 0$.

**Corollary 5.3.** Let $(R, m, K)$ be a regular local ring containing a field of prime characteristic $p > 0$ and $J \subseteq R$ be a regular parameter ideal. For any ideal $I \subseteq R$ and any $i \geq 0$, the module $H^i_I(J)$ has finitely many associated primes.

**References**


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