

Physics 390, Midterm Exam, Solutions, 14-Feb-07

Name (please print): _____

UM ID: _____

This is a closed book exam lasting 80 minutes. You may use a calculator.

A sheet with useful constants and equations is attached. Do all 11 problems and show your work clearly. There are 110 points possible. Not all problems are worth the same number of points.

There is also one question for 10 extra credit points. Do it, if you have time.

Total: _____

Extra Credit: _____

1. (6 pts) An electron is described by a wavepacket whose length is about 0.1 nm. What is the inherent uncertainty in the speed of this electron?

Solution: The Uncertainty Principle gives us

$$\Delta p_x \simeq \frac{\hbar}{\Delta x}.$$

Assuming that the speed of the electron is non relativistic, we get

$$v_x = \frac{\Delta p_x}{m} = \frac{\hbar}{\Delta x m} = \frac{1.06 \times 10^{-34}}{(10^{-10} \text{ m})(9.11 \times 10^{-31} \text{ kg})} = \underline{1.2 \times 10^6 \text{ m/s}}$$

This problem is related to problem 5-30 from the homework.

2. (10 pts) One of the classical predictions for the photoelectric effect is that there should be a time delay between turning on the light and the emission of electrons. The goal here is to estimate this time delay.

- (a) (3 pts) Assume that a light point source emits a total power of 50 Watts (50 J/s). Calculate the energy flux from this light source, in units of eV/m²s, at the surface of a metal one meter from the lamp.
- (b) (3 pts) Assuming some reasonable size for the atom, find the energy per unit time incident on a surface atom.
- (c) (3 pts) If the work function is 5.0 eV, how long does it take for this much energy to be absorbed, assuming that all the energy hitting the atom is absorbed?
- (d) (1 pt) Is such a time delay detectable?

Solution:

$$(a) \quad I = \frac{P}{4\pi R^2} = \frac{50 \text{ W}}{4\pi(1 \text{ m})^2} \left(\frac{1}{1.602 \times 10^{-19} \text{ J/eV}} \right) = \underline{2.48 \times 10^{19} \text{ eV/m}^2 \text{ s}}.$$

(b) Take an atom to have a size of 1 Å = 1 × 10⁻¹⁰ m. It would have an area of (π/2) × 10⁻²⁰ m². Such an atom, exposed to that light would absorb about

$$\frac{dW}{dt} = IA = (2.48 \times 10^{19} \text{ eV/m}^2 \cdot \text{s})(0.5 \pi \times 10^{-20} \text{ m}^2) = \underline{0.4 \text{ eV/s}}.$$

(c) The time to overcome the 5.0 eV work function would then be

$$t = \frac{\phi}{dW/dt} = \frac{5.0 \text{ eV}}{0.4 \text{ eV/s}} = \underline{12.5 \text{ s}}.$$

(d) A 12.5 s time delay is easy to detect. This is why it was surprising why the photoelectric effect occurred *without* any time delay.

This problem is similar to problem 3-54 from the homework.

3. (8 pts) A particle is trapped in a one-dimensional infinite square well. In the ground state the energy of the particle is 2.5 eV. The particle can jump to a higher level by absorbing a photon with the proper energy. Which of the following photons can be absorbed: 6 eV, 12 eV, 20 eV, 22.5 eV or 24 eV? (show your derivation).

Solution: The energy levels are given by

$$E_n = E_1 n^2 = 2.5 n^2.$$

This leads to the following transitions for the first three excited states:

$$\begin{aligned} E_2 &= 10 \text{ eV} & \Rightarrow & \Delta E = 7.5 \text{ eV} \\ E_3 &= 22.5 \text{ eV} & \Rightarrow & \Delta E = 20.0 \text{ eV} \\ E_4 &= 40 \text{ eV} & \Rightarrow & \Delta E = 37.5 \text{ eV} \end{aligned}$$

Thus, only the 20 eV photon can be absorbed, when the particle is in the ground state.

This problem is similar to problem 6-50 from the homework.

4. (8 pts) In repeat of the Davisson and Germer electron diffraction experiment a beam of electrons with kinetic energy of 50 eV are fired at a clean surface of Nickel.

- (a) (3 pts) What is the wavelength of these electrons?
- (b) (5 pts) If the Ni atoms are arranged in a regular cubic lattice with a spacing of 0.45 nm, what is the largest angle at which a strong signal of scattered electrons will be seen?

Solution:

(a) The wavelength is given by

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{2mc^2 K_e}} = \frac{1240 \text{ eV nm}}{\sqrt{2 (5.11 \times 10^5 \text{ eV})(50 \text{ eV})}} = \underline{0.174 \text{ nm}}.$$

(b) The Bragg condition is $n\lambda = D \sin \theta$, or

$$\sin \theta = \frac{n\lambda}{D} = n(0.174 \text{ nm}/0.45 \text{ nm}) = n(0.386).$$

For this, the largest allowed value of n is 2. So we have

$$\theta = \sin^{-1}(0.771) = \underline{50.4^\circ}.$$

This problem is similar to problem 3 from the practice exam and problem 5-15 from the homework.

5. (14 pts) When an X-ray photon having a wavelength of 0.0800 nm collides with a free electron, initially at rest, its wavelength increases to 0.0821 nm after the collision.

- (a) (4 pts) What is the kinetic energy of the recoiling electron?
- (b) (4 pts) At what angle, relative to its initial direction, did the photon scatter?
- (c) (6 pts) At what angle, relative to the initial direction of the photon, did the electron scatter?

Solution:

(a) Using energy conservation, the energy gained by the electron must equal that lost by the photon. The energy is

$$K_e = E - E' = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = \frac{1240 \text{ eV nm}}{0.08 \text{ nm}} - \frac{1240 \text{ eV nm}}{0.0821 \text{ nm}} = \underline{396 \text{ eV}}.$$

(b) From the Compton scattering formula

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos \theta) \Rightarrow \cos \theta = 1 - \frac{mc^2}{hc}(\lambda' - \lambda) = 1 - \frac{511 \times 10^3 \text{ eV}}{1240 \text{ eV} \cdot \text{nm}}(0.0021 \text{ nm}) = 1 - 0.865.$$

Thus, $\theta = \cos^{-1} 0.135 = \underline{82.3^\circ}$.

(c) Since the electron's kinetic energy is much smaller than its rest mass of 511 keV, we can use the non-relativistic expression for the momentum $p = \sqrt{2mc^2 K_e}/c$. From momentum conservation we then get

$$\frac{h}{\lambda'} \sin 82.3^\circ = p_e \sin \phi = \frac{1}{c} \sqrt{2mc^2 K_e} \sin \phi.$$

Thus

$$\sin \phi = \frac{(hc) \sin 82.3^\circ}{\lambda' \sqrt{2mc^2 K_e}} = \frac{(1240 \text{ nm} \cdot \text{eV}) \sin 82.3^\circ}{0.0821 \text{ nm} \sqrt{2 (5.11 \times 10^5 \text{ eV})(396 \text{ eV})}} = 0.744.$$

Or $\phi = \underline{48.1^\circ}$.

This problem is similar to problem 3-36 from the homework.

6. (6 pts) A physicist is performing the Rutherford scattering experiment by scattering a beam of alpha particles from a 4.0×10^{-6} m thick gold foil. When she places her detector at an angle of 30° relative to the incident beam, she observes that an average of 1000 particles per second are scattered into her detector. What counting rate would she observe if she moved the detector to an angle of 50° ?

Solution:

The angular dependence is $1/\sin^4(\theta/2)$. So the new rate will be

$$R = (840/s) \frac{\sin^4(30^\circ/2)}{\sin^4(50^\circ/2)} = \underline{141/s}.$$

This problem is related to problem 4-6 from the homework.

7. (8 pts)

- (a) (4 pts) What is the longest wavelength photon emitted in the Balmer series. Remember, the lowest energy level for the Balmer series for the hydrogen atom is $n_f = 2$.
- (b) (4 pts) Does this wavelength fall in the visible spectrum. If so, what color does it represent?

Solution:

(a) The longest wavelength corresponds to the smallest transition energy, thus $n_i = 3$, and $n_f = 2$. From the Balmer formula,

$$\frac{1}{\lambda_{min}} = R_\infty \left(\frac{1}{2^2} - \frac{1}{3^2} \right) = \frac{5}{36} R_\infty \quad \text{or} \quad \lambda_{min} = \underline{656.3 \text{ nm}}.$$

(b) The visible spectrum ranges from about 400 nm (violet) to 700 nm (red). Therefore, the longest wavelength photon emitted in the Balmer series is red.

This problem is very similar to the example we did in class on 19-Jan.

8. (8 pts) If the ground state energy of an electron in a box were of the same magnitude as hydrogen in the ground state,

- (a) (4 pts) How would the width of the box compare to the Bohr radius?
- (b) (4 pts) What conclusion can you draw about the Coulomb potential vs the infinite box potential?

Solution:

(a) For a particle in a box, the ground state energy is

$$E = \frac{\hbar^2 \pi^2}{2mL^2} \implies L = \frac{\hbar \pi}{\sqrt{2mE}} = \frac{\hbar c \pi}{\sqrt{2mc^2 E}}.$$

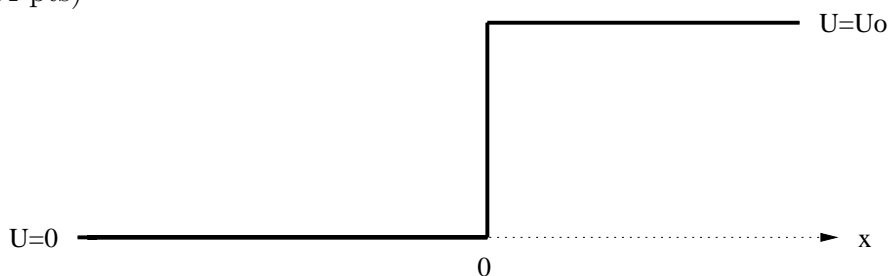
Taking the electron mass for m and $E=13.6$ eV, we have

$$\begin{aligned} L &= \frac{\pi(197 \text{ eV} \cdot \text{nm})}{\sqrt{2(511 \times 10^3 \text{ eV})(13.6 \text{ eV})}} \\ &= 0.17 \text{ nm}. \end{aligned}$$

(b) This is pretty close (within about a factor of 3) of the Bohr radius, $a_0 = 0.0529$ nm. The Bohr radius is smaller than the length scale that emerges from this particle in a box calculation. This says that the Coulomb potential binds the electron more tightly than the crude approximation of a particle in a box would suggest.

This problem is the same as question 1 in homework 3.

9. (14 pts)



A beam of particles is incident from the left upon the potential discontinuity shown in the Figure above. The time independent wave function appropriate for the region to the left of the discontinuity ($x < 0$) is

$$\psi_1(x) = 400e^{ik_1x} + Be^{-ik_1x},$$

where B is an undetermined constant. To the right of the discontinuity ($x > 0$), the wavefunction is

$$\psi_2(x) = 600e^{ik_2x}.$$

- (a) (9 pts) Use a boundary condition to find the fraction of particles (in %) which are transmitted at the discontinuity.
- (b) (5 pts) Judging from the form of wavefunction ψ_2 , what can you say about the energy of the particles incident from the left? Is it larger, equal, or smaller than the potential energy U_0 ? Be sure to indicate the reasoning you used to obtain your choice; don't just guess the answer.

Solution:

(a)

$$\psi_1(0) = \psi_2(0) = 400 + B = 600 \quad \text{or} \quad B = 200$$

$$R = \left| \frac{B}{A} \right|^2 = \left| \frac{200}{400} \right|^2 = 0.25$$

$$T = 1 - R = \underline{0.75}.$$

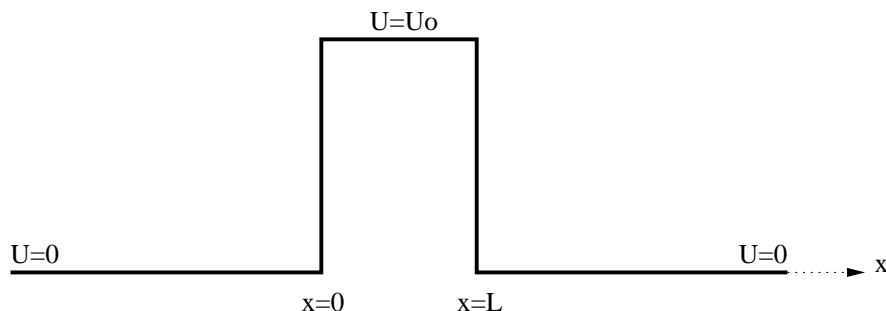
(b) There are two possible answers:

- (i) $\psi_2(x) = 600e^{ik_2x}$ is a plane wave \rightarrow represents a particle moving to the right
 $\Rightarrow E > U_0$
if $E < U_0$ $\psi_2(x)$ is of form $e^{-\alpha x}$ (exponentially decaying wave function)

- (ii) if $R \neq 1 \rightarrow E > U_0$ (from a)
if $E < U_0$ then particle is always reflected ($R = 1$)

This problem is related to problem 6-42 from the homework.

10. (16 pts) Consider the square barrier as shown in the sketch below. Assume that a right-moving particle (represented by a plane wave) is sent in from $x = -\infty$:



- (a) (6 pts) Including six undetermined constants, write down the wave functions for the three regions $x < 0$, $0 < x < L$, $x > L$, if $E < U_0$.
- (b) (4 pts) One of the coefficients must be set to zero. Which one? Why?
- (c) (6 pts) Sketch the wave function for the case where $E < U_0$. Note: Do not attempt to apply the boundary conditions explicitly; just show the expected form of the wave function!

Solution:

(a)

$$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x}$$

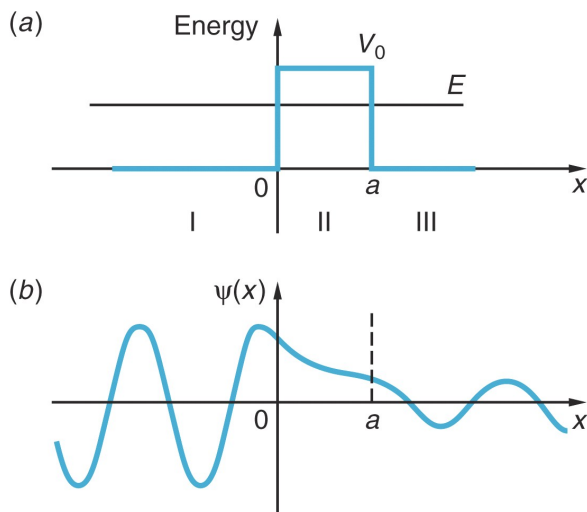
$$\psi_2 = Ce^{\alpha x} + De^{-\alpha x}$$

$$\psi_3 = Ee^{ik_1x} + Fe^{-ik_1x}$$

(b)

$F=0$ because term with F describes particle incident from the right

(c)



This problem is identical to Eq. 6.74 and Fig 6-27 in book.

11. (12 pts) It is important for physicists to have a good “ballpark” knowledge of physical quantities. Please give an approximate value for each of the items below (2 points each). You will receive credit if your answer lies within a factor of two of the correct answer.

(a) Energy of a photon of visible light.

Energies range from about 1.6 eV (dark red) to 2.5 eV (dark blue).

(b) Wavelength of a typical K_α x-ray photon.

These are given by Moseley’s law expressed in terms of wavelength, $\lambda = \frac{4}{3R_\infty(Z-1)^2}$. For a typical element in the middle of the periodic table, say $Z = 45$, this gives $\lambda = 0.06$ nm. Answers in the range 10 nm – 0.01 nm are generally OK.

(c) Diameter of an atom.

$$1 \text{ \AA} = 10^{-10} \text{ m.}$$

(d) Diameter of a nucleus.

$$1 \text{ fm} = 10^{-15} \text{ m.}$$

(e) Range of spectral lines (in nm) for Balmer series on H atom.

365 nm (UV) to 656 nm (visible: red)

(f) The lowest energy level in a harmonic oscillator.

$$E_0 = \frac{1}{2} \hbar\omega$$

Bonus Question (10 pts) An electron occupies the ground state of a one-dimensional infinite square well between $0 < x < L$. If you measure the position of the electron, what is the probability that you will find it between 0 and $L/4$?

Solution:

The ground-state wave function is

$$\psi(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L}.$$

The probability to find the particle between 0 and $L/4$ is found by squaring the wave function and integrating:

$$P = \int_0^{L/4} \psi^*(x)\psi(x) dx = \frac{2}{L} \int_0^{L/4} \sin^2 \frac{\pi x}{L} dx.$$

Using the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, this becomes

$$\begin{aligned} P &= \frac{1}{L} \left(\frac{L}{4} - \int_0^{L/4} \cos \frac{2\pi x}{L} dx \right) = \frac{1}{4} - \frac{1}{2\pi} \int_0^{\pi/2} \cos u du \\ &= \frac{1}{4} - \frac{1}{2\pi} \sin u \Big|_0^{\pi/2} \\ &= \frac{1}{4} - \frac{1}{2\pi} \\ &\approx \underline{0.0908}. \end{aligned}$$

Useful constants and equations

$$\begin{aligned}
 e &= 1.602 \times 10^{-19} \text{ C} & \frac{1}{4\pi\epsilon_0} &= 9.0 \times 10^9 \text{ N m}^2/\text{C}^2 & \frac{e^2}{4\pi\epsilon_0} &= 1.44 \text{ eV nm} \\
 c &= 3.00 \times 10^8 \text{ m/s} & h &= 6.626 \times 10^{-34} \text{ J s} & &= 4.136 \times 10^{-15} \text{ eV s} & \hbar &= \frac{h}{2\pi} \\
 hc &= 1240 \text{ eV nm} & \hbar c &= 197.3 \text{ eV nm} & 1 \text{ eV} &= 1.602 \times 10^{-19} \text{ J} & R_\infty &= 1.097 \times 10^7 \text{ m}^{-1} \\
 m_e &= 9.11 \times 10^{-31} \text{ kg} & &= 511 \text{ keV}/c^2 & a_0 &= 0.0529 \text{ nm} & E_0 &= -13.6 \text{ eV} \\
 \sigma &= 5.670 \times 10^{-8} \text{ W/m}^2\text{K}^4
 \end{aligned}$$

Electromagnetic Waves :

$$\begin{aligned}
 c &= \lambda\nu = \omega/k & k &= 2\pi/\lambda \\
 E &= \hbar\omega = pc = hc/\lambda \\
 D \sin \theta &= n\lambda \quad (\text{Bragg condition})
 \end{aligned}$$

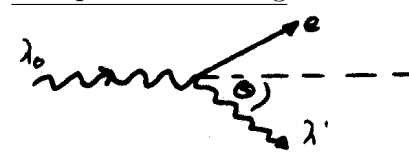
for plane waves :

$$\begin{aligned}
 \vec{E} &= \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t) \\
 \vec{B} &= \vec{B}_0 \cos(\vec{k} \cdot \vec{r} - \omega t) \\
 \vec{B} &= \frac{1}{\omega} \vec{k} \times \vec{E} \quad \text{and} \quad E_0 = cB_0
 \end{aligned}$$

Particle Waves:

$$\begin{aligned}
 p &= h/\lambda = \hbar k \\
 E &= h\nu = \hbar\omega = \sqrt{(pc)^2 + (mc^2)^2} \\
 v_{\text{phase}} &= \omega/k \\
 v_{\text{group}} &= d\omega/dk|_{k_0} = [v_{\text{phase}} + k dv_{\text{phase}}/dk]_{k_0}
 \end{aligned}$$

Compton Scattering:



$$\lambda' = \lambda_0 + \frac{h}{mc}(1 - \cos \theta)$$

Blackbody radiation:

$$R = \sigma T^4 \quad \lambda_{\text{max}}T = 2.898 \times 10^{-3} \text{ mK}$$

Photoelectric effect:

$$E = h\nu = K_e + \Phi = eV_s + \Phi$$

Heisenberg:

$$\Delta p_x \Delta x \sim \hbar \quad \Delta E \Delta t \sim \hbar$$

Schrödinger Equation:

$$\begin{aligned}
 \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U \Psi &= i\hbar \frac{\partial \Psi}{\partial t} \quad \text{for } \Psi = \psi(x) e^{-i\frac{E}{\hbar}t} \quad \text{we obtain} \quad -\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + U(x) \psi(x) = E\psi(x) \\
 P dx &= |\psi|^2 dx \quad \text{and} \quad |\psi|^2 = \psi \psi^*
 \end{aligned}$$

Infinite square well:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) \quad E_n = \frac{\pi^2 \hbar^2}{2mL^2} n^2 \quad n = 1, 2, 3, 4$$

Harmonic Oscillator:

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega \quad n = 0, 1, 2, 3$$

Operators:

$$[p_x] = \frac{\hbar}{i} \frac{\partial}{\partial x} \quad [E] = i\hbar \frac{\partial}{\partial t} \quad [KE] = \frac{[p_x^2]}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

Reflection coefficient:

$$R = \left|\frac{B}{A}\right|^2 \quad \text{for wave } \psi_1 = Ae^{ikx} + Be^{-ikx} \text{ incident from left on potential step}$$

with: $R + T = 1$ where T is transmission coefficient

Hydrogen Atom:

$$E = \frac{-13.6 \text{ eV}}{n^2} \quad \text{with} \quad n = 1, 2, 3, \dots \quad l < n \quad -l \leq m \leq +l$$

Generalized Balmer Formula:

$$\frac{1}{\lambda} = R_\infty \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

Rutherford Scattering:

$$N(\theta) = k \left(\frac{z \cdot Z}{2K_e}\right)^2 \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{\sin^4(\theta/2)} \quad \text{with} \quad k (= \text{const}) \propto nt$$

θ = scattering angle n = particle density and t = foil thickness