March 16, 2007

Physics 390: Homework set #5
Solutions

Reading: Tipler & Llewellyn, Chapter 8 (1-5), Chapter 9 (4-6), Chapter 10 (2-8)

Questions:

1. It is generally more convenient whenever possible to use the Maxwell-Boltzmann distribution, rather than quantum statistics. Under what conditions can quantum systems be described by classical statistics?

Solution: Generally, classical statistics are appropriate whenever the particles are far enough apart that we can regard them as distinguishable. This occurs when the average separation is greater than the de Broglie wavelength, or

$$\left(\frac{N}{V}\right)\frac{h^3}{(3mkT)^{3/2}} \ll 1.$$

See the discussion on pp. 358-9 for further details.

2. Estimate the mean kinetic energy of the "free" electrons in a metal if they obeyed Maxwell-Boltzmann statistics. How does this compare with the actual result from applying Fermi-Dirac statistics? Why is there such a difference?

Solution: According to classical statistics, a free electron with three translational degrees of freedom should obey the equipartition theorem. At room temperature of 300 K, the average energy would then be

$$E = \frac{3}{2}kT = \frac{3}{2}(8.61 \times 10^{-5}\ \text{eV/K})(300\ \text{K}) = 0.039\ \text{eV}.$$

From Fermi-Dirac statistics, however, we find that average energies are on the order of the Fermi energy, which are typically about 5 eV. So the classical prediction is wrong by about two orders of magnitude. The difference is due to the fact that electrons obey the exclusion principle, and cannot all fit in the low-lying states near kT. So "most" of the electrons occupy higher-energy states nearer to the Fermi energy, just like "most" of the electrons in a multi-electron atom are in the outer shells.

3. Three identical, indistinguishable particles are placed into a system consisting of four energy levels with energies 1.0, 2.0, 3.0, and 4.0 eV, respectively. The total energy of the three particles is 6.0 eV. What is the average number of particles occupying each energy level, if those particles are (a) bosons, or (b) fermions?

Solution:
To solve this problem, let's distribute the particles into the energy levels such that $E_{\text{tot}} = 6$ eV. There are three configurations of doing this.

Configuration index	Configurations with $E_{\text{tot}} = 6$ eV			
	E_1	E_2	E_3	E_4
1	o	o	o	
2	oo			o
3		ooo		

(a) Since the particles are indistinguishable bosons, each of the three configurations can only be counted once. Therefore:

$$\begin{aligned} n_1 &= 3/3 = 1.00 \\ n_2 &= 4/3 = 1.33 \\ n_3 &= 1/3 = 0.33 \\ n_4 &= 1/3 = 0.33 \end{aligned}$$

With $\sum_i^4 n_i = 3.00$ as required.

(b) Since the particles are indistinguishable fermions, only one fermion can be put into each energy level (the energy levels are non-degenerate). Thus, only the first configuration is possible.
Therefore:

$$\begin{aligned} n_1 &= 1/1 = 1.00 \\ n_2 &= 1/1 = 1.00 \\ n_3 &= 1/1 = 1.00 \\ n_4 &= 0/1 = 0.00 \end{aligned}$$

With $\sum_i^4 n_i = 3.00$ as required.

Problems:	Chapter 8:	15, 22, 33, 45
	Chapter 9:	27, 35, 38
	Chapter 10:	12, 17, 22, 26

Problem 8-15: From Eqn. 8-35

$$n(E) = \frac{2\pi N}{(\pi kT)^{3/2}} E^{1/2} e^{-E/kT}.$$

The most probable kinetic energy is found at the maximum of the distribution, where

$$\frac{dn}{dE} = 0 = \frac{2\pi N}{(\pi kT)^{3/2}} \left[\frac{1}{2}E^{-1/2} + E^{1/2}\left(-\frac{1}{kT}\right)\right] e^{-E/kT} = E^{-1/2} e^{-E/kT}\left(\frac{1}{2} - \frac{E}{kT}\right).$$

The maximum corresponds to the vanishing of the last factor. The vanishing of the other two factors correspond to a minima at $E = 0$ and $E = \infty$. Therefore,

$$\frac{1}{2} - \frac{E}{kT} = 0 \quad \rightarrow \quad \underline{E = \frac{1}{2}kT.}$$

Problem 8-22: The de Broglie wavelength for an H_2 molecule is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m\langle E\rangle}} = \frac{h}{\sqrt{2m(3kT/2)}} = \frac{h}{\sqrt{3mkT}}.$$

Meanwhile, the average distance between molecules in an ideal gas is $(V/N)^{1/3}$, which we can find from the ideal gas law:

$$PV = nRT = NkT \Longrightarrow (V/N)^{1/3} = (kT/P)^{1/3}.$$

Equating this separation to the de Broglie wavelength, we have

$$(kT/P)^{1/3} = \frac{h}{\sqrt{3mkT}}.$$

Solving for T gives

$$T = \left[\frac{Ph^3}{k(3mk)^{3/2}}\right]^{2/5}.$$

Assuming that the pressure remains at 1 atm = 101,000 Pa, we have

$$T = \left[\frac{(101,000 \text{ Pa})(6.63\times 10^{-34} \text{ J}\cdot\text{s})^3}{\{3(2\times 1.67\times 10^{-27} \text{ kg})\}^{3/2}(1.38\times 10^{-23} \text{ J/K})^{5/2}}\right]^{2/5} = \underline{4.4 \text{ K}}.$$

Problem 8-33: Approximating the nuclear potential with a 3-dimensional infinite square well and ignoring the Coulomb repulsion of the protons, the energy levels for both protons and neutrons are given by

$$E_{n_x n_y n_z} = \frac{(n_x^2 + n_y^2 + n_z^2)h^2}{8mL^2}$$

The ten protons will fill in the first five levels, which are the ground state (111), the three degenerate levels of the first excited state (112, 121, 211), and one of the three degenerate levels of the second excited state (122, 212, 221). (See the figure and discussion on p. 292). So

$$E_F(\text{protons}) = E_{122} = \frac{9h^2}{8mL^2} = \frac{9(1240 \text{ Mev}\cdot\text{fm})^2}{8(1.0078u\times 931.5 \text{ MeV}/u)(2\cdot 3.15 \text{ fm})^2} = \underline{46.5 \text{ MeV}}.$$

The twelve neutrons will fill up the first six states, so they occupy the same levels as the protons, but take up two of the degenerate levels of the second excited state. So

$$E_F(\text{neutrons}) = \underline{46.5 \text{ MeV}}$$

as well.
The average energy of both the protons and neutrons is then (Eqn. 10-37)

$$\langle E\rangle = \frac{3}{5}E_F = \underline{31 \text{ MeV}}.$$

As we will see in Chapter 11, these numbers are about right for nuclear energy levels.

The solution in the book is incorrect!

Other Solution: The Fermi energy E_F of a proton or a neutron is

$$E_F = \frac{h^2}{2m}\left(\frac{3}{8\pi}\,\frac{N}{V}\right)^{2/3} = \frac{(hc)^2}{2mc^2}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{N}{V}\right)^{2/3}.$$

The protons and neutrons in the $^{22}_{10}$Ne atom occupy a volume

$$V = \frac{4\pi}{3}R^3 = \frac{4\pi}{3}(3.1\text{ fm})^3 = 124.8\text{ fm}^3.$$

Note, that the value for N is 10 for protons and 12 for neutrons, because protons and neutrons are distinguishable particles. Thus

$$E_F(p) = \frac{(1240\text{ MeV}\cdot\text{fm})^2}{2(940\text{ MeV})}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{10}{124.8}\right)^{2/3} = \underline{36.9\text{ MeV}}.$$

$$E_F(n) = \frac{(1240\text{ MeV}\cdot\text{fm})^2}{2(940\text{ MeV})}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{12}{124.8}\right)^{2/3} = \underline{41.6\text{ MeV}}.$$

thus, E_F is weighted mean of the proton and neutron $\underline{E_F = 39.5MeV}$, and

$$\langle E\rangle = \frac{3}{5}E_F = \underline{23.6\text{ MeV}}.$$

Problem 8-45:

(a) Assuming that each state is nondegenerate, so that $g_i = 1$, we need

$$N = \sum_i n_i = f_0 + f_1 = Ce^0 + Ce^{-\epsilon/kT} = C(1 + e^{-\epsilon/kT}).$$

So

$$C = \frac{N}{1 + e^{-\epsilon/kT}}.$$

(b) The average energy is

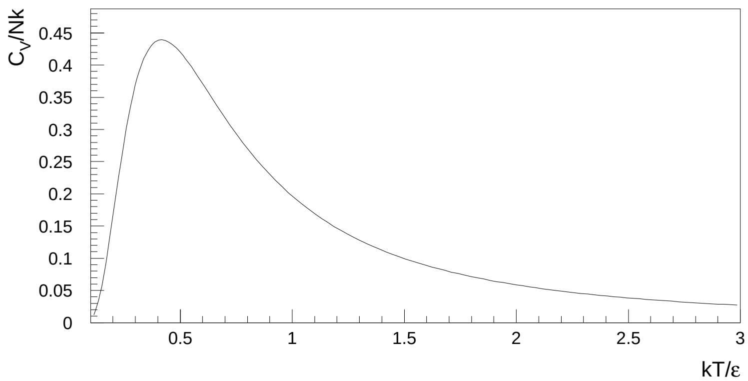
$$\langle E\rangle = \frac{1}{N}\sum_i E_i n_i = \frac{0\cdot n_0 + \epsilon n_1}{N} = \frac{\epsilon C e^{-\epsilon/kT}}{N} = \frac{N\epsilon e^{-\epsilon/kT}}{(1+e^{-\epsilon/kT})N} = \frac{\epsilon e^{-\epsilon/kT}}{1+e^{-\epsilon/kT}}.$$

As $T\to 0$, $e^{-\epsilon/kT}\to 0$, so $\langle E\rangle \to 0$. That is, all the particles are in the ground state.
As $T\to\infty$, $e^{-\epsilon/kT}\to 1$, so $\langle E\rangle\to\epsilon/2$. That is, the ground state and the excited state have equal occupancies.

(c) The heat capacity is

$$\begin{aligned} C_V &= \frac{dE}{dT} = \frac{d(N\langle E\rangle)}{dT} = \frac{d}{dT}\left(\frac{N\epsilon e^{-\epsilon/kT}}{1+e^{-\epsilon/kT}}\right) \\ &= \frac{N\epsilon^2}{kT^2}\left[\frac{-(e^{-\epsilon/kT})^2}{(1+e^{-\epsilon/kT})^2} + \frac{e^{-\epsilon/kT}}{(1+e^{-\epsilon/kT})}\right] \\ &= \underline{Nk\left(\frac{\epsilon}{kT}\right)^2\frac{e^{-\epsilon/kT}}{(1+e^{-\epsilon/kT})^2}}. \end{aligned}$$

(d) This looks like:



The heat capacity is greatest when kT is about half the size of the energy gap.

Problem 9-27: If we regard the Br atom as fixed, then the rotational inertia of the HBr molecule is

$$I = m_H r_0^2.$$

Then the characteristic rotational energy is

$$\begin{aligned} E_{0r} &= \frac{\hbar^2}{2I} = \frac{\hbar^2}{2m_H r_0^2} = \frac{(1.055 \times 10^{-34}\ \mathrm{J \cdot s})^2}{2(1.0078u)(1.66 \times 10^{-27}\ \mathrm{kg/u})(0.141\ \mathrm{nm})^2(10^{-9}\ \mathrm{m/nm})^2} \\ &= 1.67 \times 10^{-22}\ \mathrm{J} = \underline{1.04 \times 10^{-3}\ \mathrm{eV}}. \end{aligned}$$

The rotational levels are $E_\ell = \ell(\ell+1)E_{0r}$ (Eqn. 9-13) for $\ell = 0, 1, 2 \ldots$. The four lowest states have energies

$$\begin{aligned} E_0 &= 0 \\ E_1 &= 2E_{0r} = \underline{2.08 \times 10^{-3}\ \mathrm{eV}} \\ E_2 &= 6E_{0r} = \underline{6.27 \times 10^{-3}\ \mathrm{eV}} \\ E_3 &= 12E_{0r} = \underline{12.5 \times 10^{-3}\ \mathrm{eV}}. \end{aligned}$$

Problem 9-35:

(a) The total energy of a 10 MW (=10^7 J/s) pulse that lasts for 1.5 ns is

$$E = (10^7\ \mathrm{J/s})(1.5 \times 10^{-9}\ \mathrm{s}) = \underline{1.5 \times 10^{-2}\ \mathrm{J}}.$$

(b) The wavelength of the emitted light for a ruby laser is $\lambda = 694.3$ nm, so the energy per photon is

$$E_\gamma = \frac{hc}{\lambda} = \frac{1240\ \mathrm{eV \cdot nm}}{694.3\ \mathrm{nm}} = 1.786\ \mathrm{eV},$$

and the number of photons is

$$n_\gamma = \frac{1.5 \times 10^{-2}\ \mathrm{J}}{(1.876\ \mathrm{eV})(1.60 \times 10^{-19}\ \mathrm{J/eV})} = \underline{5.23 \times 10^{16}}.$$

Problem 9-38:

(a) The number of atoms in the upper state to those in the lower state is

$$\frac{n(E_2)}{n(E_1)} = \frac{e^{-E_2/kT}}{e^{-E_1/kT}} = e^{-(E_2-E_1)/kT}.$$

and

$$E_2 - E_1 = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{420 \text{ nm}} = 2.95 \text{ eV}$$

At $T = 297$ K, $kT = (8.61 \times 10^{-5} \text{ eV/K})\,(297 \text{ K}) = 0.0256$ eV, and
$n(E_2) = n(E_1)\, e^{-2.95/0.0256} = 2.5 \times 10^{21}\, e^{-115} = \underline{2 \times 10^{-29} \approx 0}$.

(b) The energy emitted in a single laser pulse is

$$\Delta E = (1.8 \times 10^{21})\,(2.95 \text{ eV/photon}) = 5.31 \times 10^{21} \text{ eV} = \underline{850 \text{ J}}.$$

Problem 10-12: The number density n of free electrons, assuming 1 electron per atom, is

$$n = \frac{\rho\, N_A}{M}.$$

Thus, for

(a) for silver $$n = \frac{(10.5 \text{ g/cm}^3)(6.022 \times 10^{23} \text{ /mo;e})}{107.87 \text{ g/mole}} = 5.86 \times 10^{22} \text{ /cm}^3.$$

(b) for gold $$n = \frac{(19.3 \text{ g/cm}^3)(6.022 \times 10^{23} \text{ /mo;e})}{196.97 \text{ g/mole}} = 5.90 \times 10^{22} \text{ /cm}^3.$$

Both results are in good agreement with the experimental values from Table 10-3.

Problem 10-17:

(a) The Fermi energy, given by Eqn. 10-35, is

$$E_F = \frac{(hc)^2}{2mc^2}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

$$\text{for Ag:} \quad E_F = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{2(511 \times 10^3 \text{ eV})}\left[\frac{3(5.86 \times 10^{28} \text{ m}^{-3})}{8\pi}\left(\frac{10^{-9} \text{ m}}{1 \text{ nm}}\right)^3\right]^{2/3} = \underline{5.50 \text{ eV.}}$$

$$\text{for Fe:} \quad E_F = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{2(511 \times 10^3 \text{ eV})}\left[\frac{3(17.0 \times 10^{28} \text{ m}^{-3})}{8\pi}\left(\frac{10^{-9} \text{ m}}{1 \text{ nm}}\right)^3\right]^{2/3} = \underline{11.2 \text{ eV.}}$$

(b) The Fermi temperature, given by Eqn. 10-38, is

$$\text{for Ag:} \quad T_F = \frac{E_F}{k} = \frac{5.50 \text{ eV}}{8.617 \times 10^{-5} \text{ eV/K}} = \underline{6.38 \times 10^4 \text{ K}}.$$

$$\text{for Fe:} \quad T_F = \frac{E_F}{k} = \frac{11.2 \text{ eV}}{8.617 \times 10^{-5} \text{ eV/K}} = \underline{13.0 \times 10^4 \text{ K}}.$$

Both results are in good agreement with the experimental values from Table 10-3.

Problem 10-22: We use Eqn. 10-44,

$$U = \frac{3}{5}NE_F + \alpha N\left(\frac{kT}{E_F}\right)kT,$$

with $\alpha = \pi^2/4$. The average energy per electron is

$$\frac{U}{N} = \frac{3}{5}E_F + \frac{\pi^2}{4}\left(\frac{kT}{E_F}\right)kT.$$

For copper, $E_F = 7.06$ eV (Table 10-3), so at $T = 0$ K we have

$$\frac{U}{N} = \frac{3}{5}E_F = \frac{3}{5}(7.06\text{ eV}) = \underline{4.236\text{ eV}}.$$

At $T = 300$ K,

$$\frac{U}{N} = \frac{3}{5}(7.06\text{ eV}) + \frac{\pi^2}{4}\left(\frac{(8.61\times10^{-5}\text{ eV/K})^2(300\text{ K})^2}{7.06\text{ eV}}\right) = \underline{4.236\text{ eV}}.$$

The difference from the value at $T = 0$ is only 0.0002 eV, a consequence of the fact that $T = 300$ K is very small compared to the Fermi temperature for Cu of 81,600 K.
The classical value for the average energy is

$$\frac{U}{N} = \frac{3}{2}kT = \underline{0.039\text{ eV}},$$

which is far too small.

Problem 10-26: The wavelength of a photon that will excite an electron from the top of the valence band to the bottom of the conduction band is

$$\lambda = \frac{hc}{E} = \frac{1240\text{ eV}\cdot\text{nm}}{1.14\text{ eV}} = 1.088\times10^3\text{ nm} = \underline{1.09\times10^{-6}\text{ m}}.$$