February 23, 2007

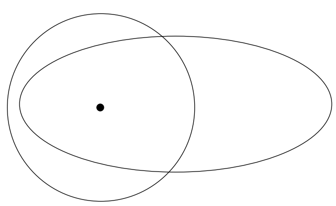
Physics 390: Homework set #4
Solutions

Reading: Tipler & Llewellyn, Chapter 7

Questions:

1. The $2s$ electron has a greater probability to be close to the nucleus than the $2p$ electron, and also a greater probability to be farther away (see Figure 7-10a). Make an analogy to classical orbits to explain how this is possible.

Solution: The s-orbits are similar to classical circular orbits, while the p-orbits, which have higher angular momentum, correspond to elliptical orbits. For particles in classical orbits with the same total energy, the one in the elliptical orbit is more likely to be found closer to the center, and further away, as illustrated here.



2. Spherical harmonics, which are eigenfunctions of angular momentum, contain the imaginary number $i = \sqrt{-1}$ (see Table 7-1). Is it all right for a function that is supposed to be associated with observable quantities to contain imaginary numbers? Why or why not?

Solution: Expectation values for observable quantities are always computed using the combination $\psi^*\psi$, which is real even if ψ itself is complex. So there is no problem with the appearance of i in the spherical harmonics.

3. Consider a penny spinning about an axis through its center at the rate of a few revolutions per second. Estimate the value of l.

Solution: Some crude estimates:

- Mass of penny: 3 g
- Radius of penny: 1 cm
- Moment of inertia of spinning penny: I'm assuming it's spinning upright, rather than like a merry-go-round. For this case the moment of interia is $\frac{1}{4}mr^2$. (If you made the other assumption, the moment of inertial only differs by a factor of two, which is unimportant here).

- Rotation rate: ≈3 Hz.

Then

$$L = I\omega = \frac{1}{4}mr^2\omega = \hbar\sqrt{\ell(\ell+1)} \approx \hbar\ell,$$

where I assumed that ℓ will be sufficiently large that $\ell \approx \ell + 1$. Solving for ℓ, we have

$$\begin{aligned} \ell &= \frac{mr^2\omega}{4\hbar} \\ &\approx \frac{(0.003 \text{ kg})(0.01 \text{ m})^2(3 \text{ Hz})(2\pi)}{4(1.05\times10^{-34} \text{ J}\cdot\text{s})} \\ &= \underline{1.3\times10^{28}} \end{aligned}$$

This very large value is why we can describe the spinning penny classically.

Problems: 1, 13, 23, 33, 34, 36, 42, 45, 68

Problem 7-1: From Eqn. 7-4

$$E_{n_1\,n_2\,n_3} = \frac{\hbar^2\pi^2}{2mL^2}\left(n_1^2+n_2^2+n_3^2\right).$$

So

$$E_{3\,1\,1} = \frac{\hbar^2\pi^2}{2mL^2}\left(3^2+1^2+1^2\right) = \underline{11E_0} \quad \text{where} \quad E_0 = \frac{\hbar^2\pi^2}{2mL^2},$$

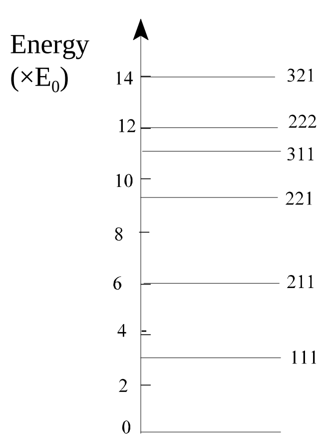
and

$$E_{2\,2\,2} = E_0(2^2+2^2+2^2 = \underline{12E_0},$$

and

$$E_{3\,2\,1} = E_0(3^2+2^2+1^2 = \underline{14E_0},$$

The first, second, third and fifth excited states are degenerate. The results are summarized in this graph:



Problem 7-13: We can express $L_x^2 + L_y^2$ as follows:

$$L_x^2 + L_y^2 = L^2 - L_z^2 = \ell(\ell+1)\hbar^2 - (m\hbar)^2.$$

For $\ell = 2$, then, $L_x^2 + L_y^2 = (6 - m^2)\hbar^2$. m can take on integer values from -2 to 2.

(a) $(L_x^2 + L_y^2)_{\text{min}} = (6 - 2^2)\hbar^2 = \underline{2\hbar^2}$.

(b) $(L_x^2 + L_y^2)_{\text{max}} = (6 - 0^2)\hbar^2 = \underline{6\hbar^2}$.

(c) $L_x^2 + L_y^2 = (6 - 1^2)\hbar^2 = \underline{5\hbar^2}$. L_x and L_y cannot be separately determined.

(d) Since ℓ cannot be larger than $n - 1$, n must be at least 3.

Problem 7-23: The normalized $n = 2, \ell = 0$ radial wavefunction for hydrogen ($Z = 1$) is (using Equation 7-33 and the result of problem 7-25)

$$\psi_{200} = \frac{1}{\sqrt{2\pi}} \frac{1}{a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}.$$

With this choice of normalization constant, the radial probability density is (see Eqn. 7-32)

$$P(r)\,dr = \psi^*\psi\, 4\pi r^2\, dr = \frac{1}{2\pi} \frac{1}{a_0^3} \left(2 - \frac{r}{a_0}\right)^2 e^{-r/a_0}\, (4\pi r^2)\, dr.$$

The interval $\Delta r = 0.02a_0$ is sufficiently small that we can treat $P(r)$ as approximately constant over the range. Then:

(a) For $r = a_0$ we have

$$P(r)\Delta r = \frac{4\pi}{2\pi} \frac{1}{a_0^3} (2-1)^2\, e^{-1} a_0^2 (0.02a_0) = 2\,(1)\, e^{-1}(0.02) = \underline{1.5 \times 10^{-2}}.$$

(b) For $r = 2a_0$,

$$P(r)\Delta r = \frac{4\pi}{2\pi} \frac{1}{a_0^3} (0) e^{-2} a_0^2 (0.02a_0) = \underline{0}.$$

Problem 7-33:

(a) For $j = 3/2$, there are $2j + 1 = 3 + 1 = 4$ possible lines. The four lines correspond to the four m_j values $-3/2,\ -1/2,\ +1/2,\ +3/2$.

(b) For $s = 0$, $j = 1$. Thus, there are $2j + 1 = 2 + 1 = 3$ possible lines. The three lines correspond to the three m_j values $-1,\ 0,\ +1$.

Problem 7-34: The spectroscopic notation is ${}^{2S+1}L_J$

For $n = 2$ ($l = 0, 1$ and $s = 1/2$): $2\,{}^2S_{1/2}$, $2\,{}^2P_{1/2}$, $2\,{}^2P_{3/2}$.

For $n = 4$ ($l = 0, 1, 2, 3$ and $s = 1/2$):
$4\,{}^2S_{1/2}$, $4\,{}^2P_{1/2}$, $4\,{}^2P_{3/2}$, $4\,{}^2D_{3/2}$, $4\,{}^2D_{5/2}$, $4\,{}^2F_{5/2}$, $4\,{}^2F_{7/2}$.

Problem 7-36:

(a) For $l = 2$, $j = l \pm 1/2$, so $j = 5/2$ or $j = 3/2$.

(b) The magnitude of the total angular momentum J is

$$\begin{aligned} J = \sqrt{j(j+1)}\hbar &= \sqrt{\frac{5}{2}(5/2+1)}\hbar = 2.96\hbar, \\ \text{or} &= \sqrt{\frac{3}{2}(3/2+1)}\hbar = 1.94\hbar, \end{aligned}$$

(c) The total angular momentum vector $\mathbf{J} = \mathbf{L} + \mathbf{S}$ has $z-$components $J_z = L_z + S_z = m_l\hbar + m_s\hbar = m_j\hbar$, where $m_j = -j, -j+1, ...j-1, j$.
So for $j = 5/2$ the $z-$components are -5/2, -3/2, -1/2, 1/2, 3/2, 5/2.
For $j = 3/2$ the $z-$components are -3/2, -1/2, 1/2, 3/2.

Problem 7-42: Neutrons have antisymmetric wave functions, so two neutrons cannot occupy the same quantum state. Ignoring spin, the minimum energy configuration occurs when one is in the $n = 1$ state and the other is in the $n = 2$ state. So

$$E = E_1 + E_2 = (1^2 + 2^2)E_1 = 5E_1,$$

where

$$E_1 = \frac{(\hbar c)^2\pi^2}{2mc^2L^2} = \frac{(197.3 \text{ MeV}\cdot\text{fm})\pi^2}{2(939.6 \text{ MeV})(2.0 \text{ fm})} = 51.1 \text{ MeV} \Longrightarrow \text{E} = 5\text{E}_1 = \underline{255 \text{ MeV}}.$$

Real neutrons, of course, have spin-1/2, so we could put two of them in the $n = 1$ state, with one neutron having spin-up and the other having spin-down.

(Problem 7-45):

(a) Chlorine has $Z = 17$: $1s^2 2s^2 2p^6 3s^2 3p^5$.

(b) Calcium has $Z = 20$: $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2$.

(c) Germanium has $Z = 32$: $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^2$.

Problem 7-68: The radial probability density for the ground state of hydrogen is (Eqns. 7-31, 7-32):

$$P(r)\,dr = 4\pi r^2 C_{100}^2 e^{-2r/a_0}\,dr = 4\pi r^2 \frac{1}{\pi}\left(\frac{1}{a_0^3}\right) e^{-2r/a_0}\,dr$$

At the edge of the proton, where $r = R_0 = 10^{-15}$ m, the exponential factor has decreased from 1 to

$$e^{-2R_0/a_0} = e^{-2(10^{-15})/0.59\times10^{-10}} = e^{-3.78\times10^{-5}} \approx 1 - 3.78\times10^{-5} \approx 1.$$

Thus to better than four significant figures, the probability of finding the electron inside the nucleus is

$$P = \int_0^{R_0} P(r)\,dr \approx \int_0^{R_0} \frac{4r^2}{a_0^3}\,dr = \frac{4}{a_0^3}\int_0^{R_0} r^2\,dr = \frac{4}{a_0^3}\left.\frac{r^3}{3}\right|_0^{R_0} = \frac{4}{3}\left(\frac{R_0}{a_0}\right)^3 = \frac{4}{3}(3.78\times10^{-5})^3 = \underline{9.0\times10^{-15}}.$$