January 26, 2007

Physics 390: Homework set #2
Solutions

Reading: Tipler & Llewellyn, Chapters 4, 5

Questions:

1. Suppose we cover one slit in the two-slit electron experiment with a very thin sheet of fluorescent material that emits a photon whenever an electron passes through. We then fire electrons one at a time at the double slit; whether or not we see a flash of light tells us which slit the electron went through. What effect does this have on the interference pattern? Why?

Solution: Performing the double slit experiment in this way would cause the interference pattern to disappear. The fluorescent detector performs a measurement that localizes the electron's position to within the width of one slit. This introduces a corresponding spread into the electron's momentum distribution, in accordance with the uncertainty principle. Since $p = h/\lambda$, there is now a corresponding spread in the wavelengths of the electrons emerging from the slits. This spread in wavelengths wipes out the interference pattern. A quantitative discussion of this problem is given in the "More" reading for 9/23.

2. In both the Rutherford theory and the Bohr theory, we neglected any wave properties of the particles. Estimate the de Broglie wavelength of an electron in a Bohr atom and compare it with the size of the atom. Estimate the de Broglie wavelength of one of Rutherfords alpha particles and compare it with the size of the nucleus. Is wave behavior expected to be important in either case?

Solution:

- For an electron in a Bohr atom: The kinetic energy of an electron in the ground state of a hydrogen atom is

$$\frac{1}{2}mv^2 = \frac{ke^2}{2a_0} = -E_1 = 13.6 \text{ eV}.$$

  This is small compared to the electron's rest mass, so we can use the nonrelativistic expression to relate its energy to its momentum and compute the de Broglie wavelength:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{hc}{\sqrt{2mc^2 E}} = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{2(511 \times 10^3 \text{ eV})(13.6 \text{ eV})}} = 0.33 \text{ nm}.$$

  This is of the same order as the size of a hydrogen atom. So yes, we expect wavelike effects to be important for atomic electrons.

- For one of Rutherford's alpha particles: these particles had kinetic energies of order 5 MeV (see Example 4-2). This is small compared to the mass of an alpha particle (about $4u = 3726$ MeV), so we can again use the nonrelativistic expression:

$$\lambda = \frac{hc}{\sqrt{2mc^2E}} = \frac{1240 \text{ MeV} \cdot \text{fm}}{\sqrt{2(3726 \text{ MeV})(5 \text{ MeV})}} = 6 \text{ fm}.$$

  This is on the order of nuclear dimensions, about 100,000× smaller than an atom. So we expect wavelike behavior to be unimportant in Rutherford scattering, which is well-described by treating the alpha particles as classical "bullets".

3. How might Moseley have measured the wavelengths of the X-rays in his experiments?

Solution: One way would have been to observe their Bragg scattering from a crystal with a known structure, such as NaCl. By observing the angle at which the Bragg condition $2d \sin\theta = m\lambda$ was satisfied, he could have determined their wavelength. From that he could have determined their energy using $E = hf = hc/\lambda$.

Problems:
Chapter 4: 3, 6, 24, 33, 45

Problem 4-3: From Eqn. 4-2,

$$\frac{1}{\lambda_{mn}} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right),$$

where $m = 1$ defines the Lyman series, for which only transitions with $n \geq 2$ are allowed. From

$$\frac{1}{164.1 \text{ nm}} = \frac{1.097 \times 10^7 \text{ m}^{-1}}{10^9 \text{ nm/m}}\left(1 - \frac{1}{n^2}\right),$$

we get

$$\frac{1}{n^2} = 1 - \frac{10^9 \text{ nm/m}}{164.1 \text{ nm}(1.097 \times 10^7 \text{ m}^{-1})} = 1 - 0.5555 = 0.4445,$$

and

$$n = \sqrt{1/0.4445} = \underline{1.5}.$$

We therefore conclude that this is not a hydrogen Lyman series transition because n is not an integer.

Problem 4-6:

(a) From Eqn. 4-5, we get the fraction f of events scattered through angles greater than θ

$$f = \pi b^2 nt.$$

For Au, the number density $n = 5.90 \times 10^{28}$ atoms/m^3 (see Example 4-2) and for this foil the thickness $t = 2.0\ \mu$m $= 2.0 \times 10^{-6}$ m. The impact parameter b is related to the angle θ by Eqn. 4-3:

$$\begin{aligned} b &= \frac{kq_\alpha Q}{m_\alpha v^2} \cot\frac{\theta}{2} = \frac{(2)(79)ke^2}{2K_\alpha} \cot\frac{90}{2} = \frac{(2)(79)(1.44 \text{ eV} \cdot \text{nm})}{2(7.0 \times 10^6 \text{ eV})} \\ &= 1.63 \times 10^{-5} \text{ nm} = 1.63 \times 10^{-14} \text{ m}. \end{aligned}$$

So

$$f = \pi(1.63 \times 10^{-14})^2(5.90 \times 10^{28})(2.0 \times 10^{-6}) = \underline{9.8 \times 10^{-5}}.$$

(b) For $\theta = 45°$,

$$\begin{aligned} b(45°) &= b(90°)\frac{\cot\frac{45°}{2}}{\cot\frac{90°}{2}} \\ &= b(90°)\frac{\tan\frac{90°}{2}}{\tan\frac{45°}{2}} \\ &= 3.92 \times 10^{-5}\ \text{nm} = 3.92 \times 10^{-14}\ \text{m} \end{aligned}$$

and $f(45°) = 5.7 \times 10^{-4}$.
For $\theta = 75°$,

$$\begin{aligned} b(75°) &= b(90°)\frac{\tan\frac{90°}{2}}{\tan\frac{75°}{2}} \\ &= 2.12 \times 10^{-5}\ \text{nm} = 2.12 \times 10^{-14}\ \text{m} \end{aligned}$$

and $f(75°) = 1.667 \times 10^{-4}$.
Therefore,

$$\Delta f(45° - 75°) = 5.7 \times 10^{-4} - 1.667 \times 10^{-4} = \underline{4.05 \times 10^{-4}}.$$

Problem 4-24:

(a) To calculate the energies of the three lowest states in positronium, the reduced mass correction to the Rydberg constant has to be applied. From Eqn. 4-26,

$$R = R_\infty\left(\frac{1}{1 + m/M}\right) = R_\infty\left(\frac{1}{2}\right) = 5.4869 \times 10^6\ \text{m}^{-1}.$$

Combining Eqns. 4-23 and 4-24, we get

$$E_n = -\frac{hcR}{n^2},$$

and

$$E_1 = -\frac{(1240\ \text{eV}\cdot\text{nm})(5.4869 \times 10^6\ \text{m}^{-1})(10^{-9}\ \text{m/mm})}{(1)^2} = \underline{-6.804\ \text{eV}}.$$

Similarly, $E_2 = \underline{-1.701\ \text{eV}}$, and $E_3 = \underline{-0.756\ \text{eV}}$.

(b) The Lyman α is the $n = 2 \rightarrow n = 1$ transition, or

$$\frac{hc}{\lambda_\alpha} = E_2 - E_1 \rightarrow \lambda_\alpha = \frac{hc}{E_2 - E_1} = \frac{(1240\ \text{eV}\cdot\text{nm})}{-1.701\ \text{eV} - (-6.804\ \text{eV})} = \underline{243\ \text{nm}}.$$

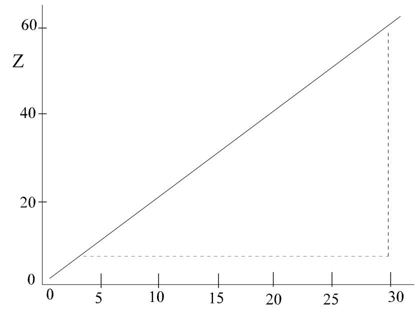
The Lyman β is the $n = 3 \rightarrow n = 1$ transition, or

$$\lambda_\beta = \frac{hc}{E_3 - E_1} = \frac{(1240\ \text{eV}\cdot\text{nm})}{0.756\ \text{eV} - (-6.804\ \text{eV})} = \underline{205\ \text{nm}}.$$

Problem 4-33: We can fill out a table in Moseley-plot-friendly format for the elements listed. (Use $f^{1/2} = (E/h)^{1/2}$.)

Element	Al	Ar	Sc	Fe	Ge	Kr	Zr	Ba
Z	13	18	21	26	32	36	40	56
E (keV)	1.56	3.19	4.46	7.06	10.98	14.10	17.66	36.35
$f^{1/2}$ (10^8 Hz)	6.14	8.77	10.37	13.05	16.28	18.45	20.64	29.62

The resulting Moseley plot looks like:



$f^{1/2}$ $(10^8\ Hz^{1/2})$

I can fit this to a line with a slope of 1.85×10^{-8} Hz$^{-1/2}$. Eyeballing Figure 4-18, I get a slope of approximately

$$\frac{30 - 13}{(15 - 7) \times 10^8} = 2.1 \times 10^{-8}\ \text{Hz}^{-1/2}.$$

These two values are in good agreement.

Problem 4-45:

(a) From Eqn. 4-20 we get for the Li^{++} ion ($Z = 3$) the following energy levels:

$$E_n = -13.6\ \text{eV}(9)/n^2 = -122.4/n^2\ \text{eV}.$$

The first three Li^{++} levels that have (nearly) the same energy as H are:

$$n = 3,\ E_3 = -13.6\ \text{eV}, \quad \text{n} = 6,\ \text{E}_6 = -3.4\ \text{eV}, \quad \text{n} = 9,\ \text{E}_9 = -1.51\ \text{eV}.$$

The Lyman α line corresponds to the $n = 6 \to n = 3$ Li^{++} transition. The Lyman β line corresponds to the $n = 9 \to n = 3$ Li^{++} transition.

(b) From Eqn. 4-26 we get the Rydberg constant for H

$$R(H) = R_\infty \left(\frac{1}{1 + 0.511\ \text{MeV}/938.8\ \text{MeV}} \right) = 1.096776 \times 10^7\ m^{-1},$$

and for Li

$$R(Li) = R_\infty \left(\frac{1}{1 + 0.511\ \text{MeV}/6535\ \text{MeV}} \right) = 1.097287 \times 10^7\ m^{-1},$$

For the Lyman α line

$$\frac{1}{\lambda} = R(H)\left(1 - \frac{1}{2^2}\right) = 1.096776 \times 10^7\ m^{-1}(10^{-9}\ \text{m/nm})(3/4) \rightarrow \lambda = 121.568\ \text{nm}.$$

For the Li^{++} equivalent

$$\frac{1}{\lambda} = R(Li)\left(\frac{1}{3^2} - \frac{1}{6^2}\right)\ Z^2 = 1.097287{\times}10^7\ m^{-1}(10^{-9}\ \text{m/nm})\left(\frac{1}{9} - \frac{1}{16}\right)(3)^2 \rightarrow \lambda = 121.512\ \text{nm}.$$

This leads to a tiny difference in wavelength of $\Delta\lambda = \underline{0.056\ \text{nm}}$.

Chapter 5: 3, 15, 18, 30, 35

Problem 5-3: The kinetic energy of nonrelativistic electrons accelerated from rest through a potential difference V_0 is

$$E_k = eV_0 = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{(hc)^2}{2mc^2\lambda^2}$$

and thus

$$V_0 = \frac{1}{e} \cdot \frac{(1240\ \text{eV} \cdot \text{nm})}{[(2)(0.511 \times 10^6\ \text{eV})(0.04\ \text{nm})]^{1/2}} = \underline{940\ \text{V}}.$$

Problem 5-15: The Bragg condition is $\sin\phi = n\lambda/D$. The 350 eV electron beam is nonrelativistic, so the de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_k}} = \frac{hc}{\sqrt{2mc^2E_k}}$$

$$= \frac{1240\ \text{eV} \cdot \text{nm}}{\sqrt{2(511 \times 10^3\ \text{eV})(350\ \text{eV})}} = \underline{0.0656\ \text{nm}}.$$

Then

$$\sin\phi = n(0.0656\ \text{nm})/(0.315\ \text{nm}) = 0.208n.$$

So we have

n	ϕ
1	12.0°
2	24.6°
3	38.6°
4	56.4°

These are the only allowed values since larger values of n would give $\sin\phi > 1$.

Problem 5-18:

(a) The phase velocity is given by

$$v_p = f\lambda = \frac{\omega}{k}.$$

From Eqn. 5-22, we get for the group velocity

$$v_{group} = \frac{d\omega}{dk} = \frac{d}{dk}(kv_p) = v_p + k\frac{dv_p}{dk}.$$

The dispersive term can be expressed as (using $k = 2\pi/\lambda$)

$$\frac{dv_p}{dk} = \frac{dv_p}{d\lambda}\frac{d\lambda}{dk} = \frac{dv_p}{d\lambda}\left(-\frac{2\pi}{k^2}\right) = \left(-\frac{\lambda}{k}\right)\frac{dv_p}{d\lambda}$$

and therefore

$$\underline{v_g = v_p - \lambda\,\frac{dv_p}{d\lambda}}.$$

(b) The index of refraction of light in glass decreases as λ increases (shorter wavelengths are refracted more than longer wavelengths). Since $n = c/v_p$, v_p decreases as λ decreases. Further, $dn/d\lambda$ and $dv_p/d\lambda$ have opposite signs, so that $dv_p/d\lambda > 0$, since $dn/d\lambda < 0$. Thus $v_{group} < v_{phase}$.

Problem 5-30: From Eqn. 5-29,

$$\Delta x\,\Delta p \approx \hbar.$$

For a particle with an uncertainty in position equal to its de Broglie wavelength,

$$\lambda\,\Delta p \approx \hbar \qquad \text{or} \qquad \Delta p \approx \hbar/\lambda \qquad \text{or} \qquad \Delta p \approx h/\lambda.$$

Because $\lambda = h/p$, and thus $p = h/\lambda$, the uncertainty in momentum Δp is

$$\underline{\Delta p \approx p}.$$

Problem 5-35: The minimum lifetime of a state with an energy uncertainty of 1 eV is (Eqn. 5-30)

$$\Delta E\,\Delta t \approx \hbar.$$

Thus

$$\tau \approx \Delta t \approx \frac{\hbar}{\Delta E} = \frac{1.055 \times 10^{-34}\ \text{Js}}{(1\ \text{eV})(1.602 \times 10^{-19}\ \text{J/eV})} = \underline{6.6 \times 10^{-16}\ \text{s}}.$$