January 17, 2007

Physics 390: Homework set #1
Solutions

Reading: Tipler & Llewellyn, Chapter 3

Questions:

1. Show that the classical wave equation

$$\frac{\partial^2 f}{\partial t^2} - c^2\frac{\partial^2 f}{\partial x^2} = 0$$

is satisfied by *any* function f that depends on x and t in the combination $u = x \pm ct$: $f(x,t) = f(u) = f(x \pm ct)$.

Solution: To see this, plug into the equation above:

$$\begin{aligned}
\frac{\partial}{\partial t} f &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial f}{\partial u}(\pm c) \\
\frac{\partial^2}{\partial t^2} f &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial u}(\pm c)\right] = (\pm c)\frac{\partial}{\partial t}\frac{\partial f}{\partial u} \\
&= (\pm c)\frac{\partial}{\partial u}\frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = (\pm c)\frac{\partial^2 f}{\partial u^2}(\pm c) \\
&= c^2\frac{\partial^2 f}{\partial u^2}
\end{aligned}$$

similarly,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{\partial^2 f}{\partial u^2}(1) = \frac{\partial^2 f}{\partial u^2}$$

and,

$$\begin{aligned}
\frac{\partial^2 f}{\partial t^2} - c^2\frac{\partial^2 f}{\partial x^2} &= c^2\frac{\partial^2 f}{\partial u^2} - c^2\frac{\partial^2 f}{\partial u^2} \\
&= 0 \quad \text{identically!}
\end{aligned}$$

2. Planck's constant is $h = 6.626 \times 10^{-34}$ J·s. What familiar physical quantity from classical mechanics also has dimensions of J·s?

Solution: Angular momentum also has dimensions of J·s. We will see that Planck's constant is closely related to quantization of angular momentum.

3. In what region of the electromagnetic spectrum does the blackbody radiation from a roomtemperature object peak? What sorts of problems would we have if our eyes were sensitive in this region?

Solution: Room temperature is about 290 K. Using the Wien displacement law, we have

$$\begin{aligned}\lambda_{\max} &= \frac{2.898 \times 10^{-3} \text{ m} \cdot \text{K}}{290 \text{ K}} \\ &= 1 \times 10^{-5} \text{ m}.\end{aligned}$$

This is in the infrared region of the spectrum. If our eyes were sensitive in this region, we would have difficulty distinguishing individual objects because all room-temperature objects would be the same "color".

4. The Compton scattering formula suggests that objects viewed from different angles should reflect light of different wavelengths. Why don't we observe a change in color of objects as we change the viewing angle?

Solution: This effect does indeed take place, but it is so tiny that it is not noticeable to our eyes. For example, the wavelength of green light from a mercury vapor streetlight is $\lambda = 546.1$ nm. The maximum Compton shift of this light would occur when it's backscattered at 180°. In this case, the shifted wavelength is

$$\begin{aligned}\lambda' &= \lambda + \frac{hc}{m_e c^2}(1 - \cos\theta) \\ &= 546.1 \text{ nm} + (0.002426 \text{ nm})(2) \\ &= 546.1002 \text{ nm}.\end{aligned}$$

Such a minuscule shift is entirely undetectable.

Problems: 3, 30, 36, 45, 49, 54

Problem 3:

(a) From Eqn. 2-10 (using v for the velocity rather than u as in the text, to avoid confusing it with the energy density spectral distribution function) we find that the total energy E of an electron with a kinetic energy of 5×10^4 eV is 0.561 MeV. From Eqns. 3-4, 2-34 and 2-31 ,

$$v = \frac{\mathcal{E}}{\mathcal{B}}, \qquad \frac{v}{c} = \frac{pc}{E}, \qquad \text{and} \qquad pc = \sqrt{E^2 - (mc^2)^2}$$

we find that

$$pc = \sqrt{(0.561 \text{ MeV})^2 - (0.511 \text{ MeV})^2} = 0.2315 \text{ MeV}$$

$$\frac{v}{c} = \frac{0.2315 \text{ MeV}}{0.561 \text{ MeV}} = 0.41$$

and therefore, for the magnetic field B:

$$\mathcal{B} = \frac{2 \times 10^5 \text{ V/m}}{0.41\, c} = 1.63 \times 10^{-3} \text{ T} = \underline{16.3 \text{ G}}$$

Problem 30: Using Eqn. 3-36,

(1) we get

$$0.95 = \frac{h}{e}\left(\frac{c}{435.8 \times 10^{-9}\ \mathrm{m}}\right) - \frac{\phi}{e}$$

(2) and

$$0.38 = \frac{h}{e}\left(\frac{c}{546.1 \times 10^{-9}\ \mathrm{m}}\right) - \frac{\phi}{e}$$

Subtracting (2) from (1), we get

$$0.57 = \frac{hc}{e\ 10^{-9}}\left(\frac{1}{435.8} - \frac{1}{546.1}\right)$$

Solving this equation for h yields: $\underline{h = 6.56 \times 10^{-34}\ \mathrm{J \cdot s}}$. Substituting h into either (1) or (2) and solving for ϕ/e yields: $\underline{\phi/e = 1.87\ \mathrm{eV}}$. The threshold frequency is given by $hf/e = \phi/e$ or

$$f = \left(\frac{\phi}{e}\right)\left(\frac{e}{h}\right) = \frac{(1.87\ \mathrm{eV})(1.60 \times 10^{-19}\ \mathrm{C})}{6.56 \times 10^{-34}\ \mathrm{J \cdot s}} = \underline{4.57 \times 10^{14}\ \mathrm{Hz}}$$

Problem 36: From Eqn. 3-40,

$$\lambda_2 - \lambda_1 = \frac{h}{mc}(1 - \cos\phi) = 0.00243\ \mathrm{nm}(1 - \cos 110^\circ) = 0.00326\ \mathrm{nm} = 3.26 \times 10^{-12}\ \mathrm{m}.$$

The wavelength of the initial photon is

$$\lambda_1 = \frac{hc}{E_1} = \frac{1240\ \mathrm{nm \cdot eV}}{511 \times 10^3\ \mathrm{eV}} = 2.43 \times 10^{-12}\ \mathrm{m}.$$

From Eqn. 3-40,

$$\lambda_2 = \lambda_1 + 3.26 \times 10^{-12}\ \mathrm{m} = 5.69 \times 10^{-12}\ \mathrm{m}.$$

and therefore,

$$E_2 = \frac{hc}{\lambda_2} = \frac{1240\ \mathrm{nm \cdot eV}}{5.69 \times 10^{-3}\ \mathrm{nm}} = \underline{0.218\ \mathrm{MeV}}.$$

From energy conservation, the electron recoil energy is

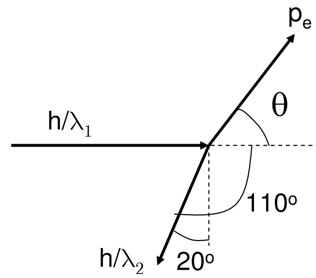
$$E_e = E_1 - E_2 = 0.511\ \mathrm{MeV} - 0.218\ \mathrm{MeV} = \underline{0.293\ \mathrm{MeV}}.$$

The recoil electron momentum makes an angle θ with the direction of the initial photon
From momentum conservation

$$\frac{h}{\lambda_2}\cos 20^\circ = p_e \sin\theta = \frac{\sqrt{E^2 - (mc^2)^2}}{c}\sin\theta.$$

Since the total relativistic energy of the electron is $E = 0.293\ \mathrm{MeV} + 0.511\ \mathrm{MeV} = 0.804\ \mathrm{MeV}$, we get

$$\begin{aligned} \sin\theta &= \frac{(hc)\ \cos 20^\circ}{\lambda_2\sqrt{E^2 - (mc^2)^2}} \\ &= \frac{(1240\ \mathrm{nm \cdot eV})\ \cos 20^\circ}{0.00569\ \mathrm{nm}\sqrt{(0.804\ \mathrm{MeV})^2 - (0.511\ \mathrm{MeV})^2}\ (10^6\ \mathrm{eV/MeV})} \\ &= 0.330. \end{aligned}$$



Or $\underline{\theta = 19.3^\circ}$.

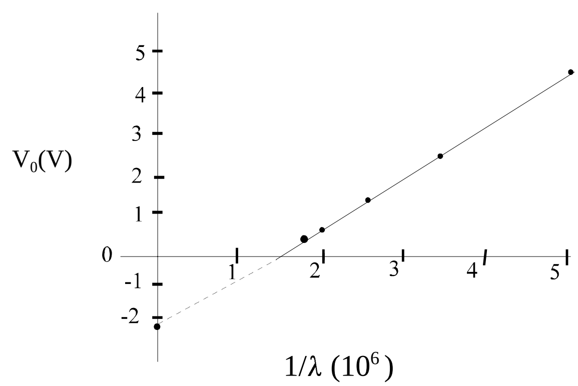
Problem 45: The photoelectric effect equation is

$$eV_0 = \frac{hc}{\lambda} - \phi.$$

So let's express the data in terms of $1/\lambda$:

$1/\lambda$ (10^6/m)	5.0	3.3	2.5	2.0	1.7
V_0 (V)	4.20	2.06	1.05	0.41	0.03

A plot of these data looks like:



I can fit these data to a straight line with a slope of 1.30×10^{-6} V·m and a y-intercept of -2.2 V.

(a) The work function, which is the vertical intercept, is $\underline{\phi = 2.2 \text{ eV}}$.

(b) The horizontal intercept yields the threshold frequency:

$$\frac{1}{\lambda_t} = \frac{2.2 \text{ V}}{1.30 \times 10^{-6} \text{ V} \cdot \text{m}} = 1.69 \times 10^6 \text{ m}^{-1}.$$

So $f_t = c/\lambda_t = \underline{5.1 \times 10^{14} \text{ Hz.}}$

(c) The slope of the line (hc/e) yields h/e:

$$\frac{h}{e} = \frac{\text{slope}}{c} = \frac{1.30 \times 10^{-6} \text{ V} \cdot \text{m}}{3 \times 10^8 \text{ m/s}} = \underline{4.33 \times 10^{-15} \text{ V} \cdot \text{s.}}$$

This is within 5% of the accepted value, 4.13×10^{-15} V · s.

Problem 49:
Conservation of energy gives

$$E_1 + mc^2 = E_2 + E_k + mc^2$$

and therefore

$$E_k = E_1 - E_2 = hf_1 - hf_2$$

From Compton's equation, we have

$$\lambda_2 - \lambda_1 = \frac{h}{mc}\,(1 - \cos\theta)$$

thus

$$\frac{1}{f_2} - \frac{1}{f_1} = \frac{h}{mc^2}\,(1 - \cos\theta)$$

from which follows that

$$f_2 = \frac{f_1\, mc^2}{mc^2 + hf_1\,(1 - \cos\theta)}$$

Substituting this expression for f_2 into the expression for E_k (and dropping the subscript on f_1) gives

$$E_k = hf - \frac{hfmc^2}{mc^2 + hf\,(1 - \cos\theta)} = \frac{hfmc^2 + (hf)^2(1-\cos\theta) - hfmc^2}{mc^2 + hf\,(1 - \cos\theta)} = \frac{hf}{1 + \frac{mc^2}{[hf(1-\cos\theta)]}}$$

E_k has its maximum value when the photon energy change is maximum, i.e. when $\theta = \pi$, so $\cos\theta = -1$. Then

$$\underline{E_k = \frac{hf}{1 + \frac{mc^2}{2hf}}}$$

Problem 54:

(a)

$$I = \frac{P}{4\pi R^2} = \frac{1\text{ W}}{4\pi(1\text{ m})^2}\left(\frac{1}{1.602\times 10^{-19}\text{ J/eV}}\right) = \underline{4.97\times 10^{17}\text{ eV/m}^2\cdot\text{s}}.$$

(b) Let's assume that an atom occupies an area of $(0.1\text{ nm})^2$. Then

$$\frac{dW}{dt} = IA = (4.97\times 10^{17}\text{ eV/m}^2\cdot\text{s})(0.1\text{ nm})^2(10^{-9}\text{ m/nm}^2) = \underline{4.97\times 10^{-3}\text{ eV/s.}}$$

(c) The time to overcome the 2 eV work function would then be

$$t = \frac{\phi}{dW/dt} = \frac{2\text{ eV}}{4.97\times 10^{-3}\text{ eV/s}} = 403\text{ s} = 6.71\text{ minutes.}$$

This is in clear contradiction to what's observed experimentally, in which electrons are ejected almost instantaneously.