Physics 390 "Final" Exam Practice Solutions

These are a few problems comparable to those you will see on the exam. They were picked from previous exams. I will provide a sheet with useful constants and equations for the exam. The last two pages are the Formula sheets for the Final Exam.

1. (8 pts) The ^{209}Bi nucleus has a spin of 9/2.

(a) (4 pts) What is the angular momentum of this nucleus (in units of $\hbar$)?

(b) (4 tps) Suppose that it were possible to make a beam of completely ionized ^{209}Bi nuclei and pass it through a Stern-Gerlach apparatus. How many distinct "blobs" would you observe on the detector screen?

Solution:
(a) $J = \sqrt{j(j+1)}\hbar = \sqrt{\frac{9}{2} \cdot \frac{11}{2}}\hbar = \frac{\sqrt{99}}{2}\hbar = 4.97\hbar$.

(b) The magnetic field in the apparatus defines a z-axis, and there will be one "blob" for each possible projection of the total momentum vector along this axis. So there will be $(2j+1) = 10$ spots, corresponding to J_z = 9/2, 7/2, 5/2, 3/2, 1/2, -1/2, -3/2 -5/2, -7/2, -9/2.

2. (8 pts) In a scattering experiment it was found that ^{12}C has a nuclear radius of 2.7 fm. The experiment is then repeated with another, unknown element and it is found the the nuclear radius is twice as big. What is the mass number of this unknown element?

Solution: The nuclear radius of the first nucleus is given by

$$R_1 = r_0\, A_1^{1/3}$$

and

$$R_2 = r_0\, A_2^{1/3}.$$

So

$$A_2 = A_1 \left(\frac{R_2}{R_1}\right)^3 = 12\ (2^3) = \underline{96}.$$

3. (8 pts) Consider the following energetically possible transitions of an electron in an atom:

(1) $4p \longrightarrow 3p$ (2) $3d \longrightarrow 2s$ (3) $4s \longrightarrow 2p$ (4) $4d \longrightarrow 3p$

Which of these transitions is/are allowed? Explain your reasoning.

Solution:
$4p \longrightarrow 3p$: is not allowed, because it is a $\Delta l = 0$ transition.
$3d \longrightarrow 2s$: is not allowed, because it is a $\Delta l = 2$ transition.
$4s \longrightarrow 2p$: is allowed, because it is a $\Delta l = 1$ transition.
$4d \longrightarrow 3p$: is allowed, because it is a $\Delta l = 1$ transition.

4. (12 pts) The proton has a magnetic moment of $\mu_p = 8.8 \times 10^{-8}$ eV/T.

(a) (5 pts) What is the energy difference between the spin-up and spin-down orientations when a proton is placed in a magnetic field of 1.5 T?

(b) (7 tps) In magnetic resonance imaging, a person with a body temperature of 310 K is placed in a 1.5 T magnetic field. Compute the ratio of $N^{\uparrow}/N^{\downarrow}$, where $N^{\uparrow}$ is the number of protons in the spin-up state (parallel to the field) state, and $N^{\downarrow}$ is the number in the spin-down state. State clearly any assumptions you make.

Solution:
(a) The energy of a magnetic dipole in a magnetic field is $E = -\boldsymbol{\mu} \cdot \mathbf{B}$. If we use the B field to define the z-axis, then the energy of the proton is

$$E = -(\mu_p)_z \, B = -g_p m_s \mu_N \, B = -\mu_p \, B.$$

The energy difference between the spin-up and spin-down orientations is

$$\Delta E = E_{\downarrow} - E_{\uparrow} = 2\mu_p \, B = 2 \cdot (8.8 \times 10^{-8} \text{ eV/T})(1.5 \text{ T}) = \underline{2.6 \times 10^{-7} \text{ eV}},$$

with the spin-up state (parallel to the field) being the lower-energy state.

(b) If we assume that the magnetic dipoles do not interact with each other we can treat them as free dipoles. Furthermore, the protons are sufficiently far apart that we can regard them as distinguishable (by virtue of their position) and use Maxwell-Boltzmann statistics, $f_{MB} = A \, e^{-E/kT}$. There is no density of states factor to worry about, since the spin-up and spin-down states are non-degenerate. Then

$$N_{\uparrow} = N \, A \, e^{-E_{\uparrow}/kT}, \qquad N_{\downarrow} = N \, A \, e^{-E_{\downarrow}/kT}.$$

So

$$\frac{N_{\uparrow}}{N_{\downarrow}} = \frac{e^{-E_{\uparrow}/kT}}{e^{-E_{\downarrow}/kT}} = e^{\Delta E/kT} \approx 1 + \frac{\Delta E}{kT} = 1 + \frac{2.6 \times 10^{-7} \text{ eV}}{(8.617 \times 10^{-5} \text{ eV/K})(310 \text{ K})} = 1 + 9.9 \times 10^{-6}.$$

So the excess of protons in the spin-up state is only a few parts per million. But because there are so many protons in the body, this small excess is enough to be useful for MRI.

5. (8 pts) If the energy levels in an atom are filled in the sequence 1s, 2s, 2p, 3s, 3p, 4s, 3d.

(a) (5 pts) What is the atomic number of the element that is in its ground state, has all these levels filled, and all higher levels empty?

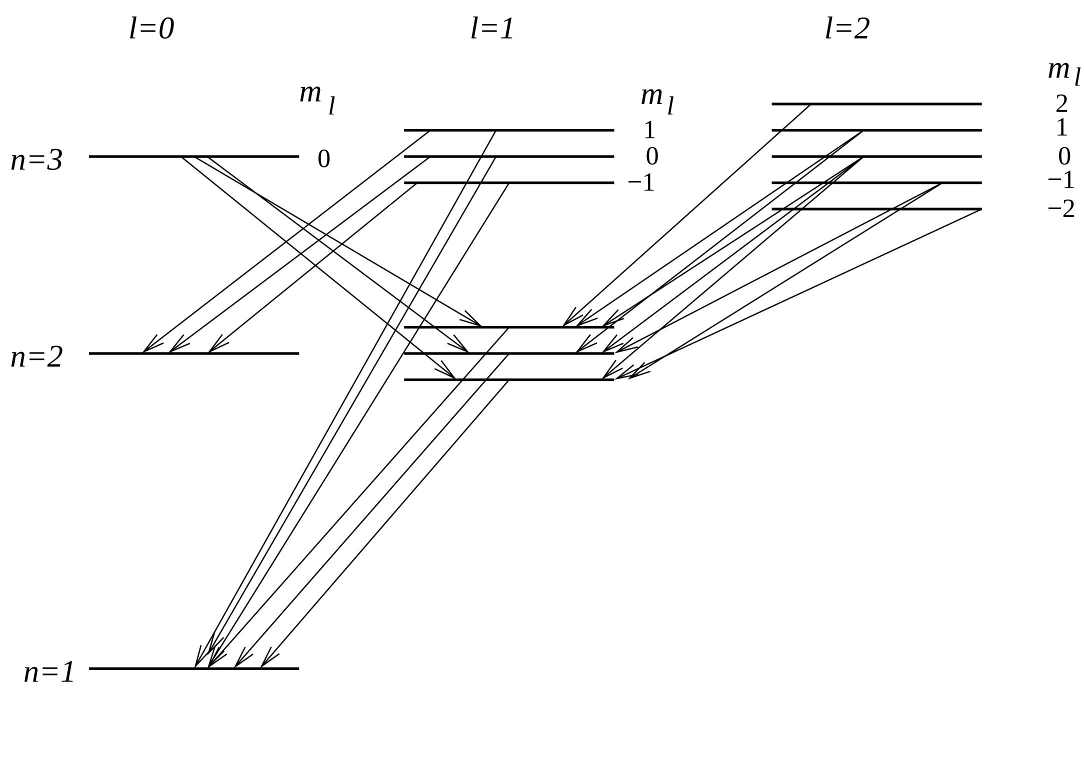
(b) (3 tps) Suppose that an electron is knocked out of the 1s level. From which of these levels could an electron drop down to fill the resulting unoccupied state? State your reasoning.

Solution:
(a) Each shell holds $2(2l+1)$ electrons. So the s shells hold 2 electrons, the p shells hold 6, and the d shell has 10. This atom has $2+2+6+2+6+2+10 = \underline{30}$ electrons. Assuming it is a neutral atom, this is the element Zn.

(b) The transitions obey the selection rule $\Delta l = \pm 1$. Since the 1s states have $l = 0$, the electron must come from an $l = 1$ state. So transitions from any of the p levels are allowed.

6. (10 pts) Sketch the $n = 1$, $n = 2$, and $n = 3$ energy levels for a hydrogen atom in a magnetic field. Indicate three possible *transitions* with solid lines. Indicate three forbidden (or at least highly suppressed) transitions with dotted lines, and state why they are forbidden.
(Ignore fine and hyperfine structure.)

Solution: The energy levels and the allowed transitions are shown below. All other transitions are forbidden because they violate the selection rules $\Delta l = \pm 1$ or $\Delta m_l = \pm 1, 0$.



7. (16 pts) When the sun runs out of fusion fuel, it will no longer be able to resist the compression of gravity and will collapse to become a white dwarf star. In this state it will have about the current mass of the Sun (2×10^{30} kg) in a sphere with a radius a bit bigger than that of the Earth (10^7 m say).

(a) (6 pts) Estimate the Fermi energy of electrons in this remnant of the Sun.

(b) (5 tps) Get an order of magnitude estimate for the **total** energy of these electrons. Hint: how does the average energy of electrons compare to the Fermi Energy? Is it roughly the same, or very different?

(c) (5 tps) Given that the pressure can be calculated from $P = -dE_{\text{total}}/dV$, estimate the pressure in this object

Solution:
(a) The Fermi energy E_F of an electron is

$$E_F = \frac{h^2}{2m_e}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3} = \frac{(hc)^2}{2m_e c^2}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{N}{V}\right)^{2/3}.$$

For N, we can assume that it is all Hydrogen ($m_H = 1.7\times10^{-27}$ kg) and obtain $N = M_{sun}/1.7\times 10^{-27}$ kg $= 1.2 \times 10^{57}$. Or if we assume it is all Helium, $N_{He} = .5\ N_H$. Putting this all together we get

$$E_F = \frac{(1240 \times 10^{-15}\ \text{MeV} \cdot \text{m})^2}{2(0.511\ \text{MeV})}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{1.2 \times 10^{57}}{(4\pi/3)(10^7)^3}\right)^{2/3} = \underline{0.16\ \text{MeV}},$$

(or 0.10 MeV for Helium) which is nearly its rest mass.

(b) Since $E_m = 3/5\ E_F$, E_{total} is roughly

$$E_{\text{total}} = \frac{3}{5}\ E_F \cdot N_{\text{electrons}} \approx \frac{3}{5} \cdot 0.16\ \text{MeV} \cdot 1.2 \times 10^{57} = \underline{1.1 \times 10^{56}\ \text{MeV}}$$

(c) The pressure is

$$P = -dE_{\text{total}}/dV = \frac{2}{3}\,\frac{3}{5}\,\frac{h^2}{2m_e}\left(\frac{3}{8\pi}\right)^{2/3} N^{5/3}\ V^{-5/3} = \underline{4.4 \times 10^{22}\ \text{N/m}^2}.$$

8. (6 pts) Protons and neutrons can interact by exchanging pions which are mesons with a rest mass of 140 MeV/c^2. What is the range of this interaction?

Solution: The range of the interaction is

$$R = \frac{\hbar c}{mc^2} = \frac{197.3\ \text{MeV} \cdot \text{fm}}{140\ \text{MeV}} = \underline{1.4\ \text{fm}}.$$

Useful constants and equations

$e = 1.602 \times 10^{-19}\ C$ $\quad \frac{1}{4\pi\epsilon_0} = 9.0 \times 10^9\ N\,m^2/C^2$ $\quad \frac{e^2}{4\pi\epsilon_0} = 1.44\ eV\,nm$

$c = 3.00 \times 10^8\ m/s$ $\quad h = 6.626 \times 10^{-34}\ J\,s = 4.136 \times 10^{-15}\ eV\ s$ $\quad \hbar = \frac{h}{2\pi}$

$hc = 1240\ eV\ nm$ $\quad \hbar c = 197.3\ MeV\ fm$ $\quad 1\ eV = 1.602 \times 10^{-19}\ J$ $\quad R_\infty = 1.097 \times 10^7\ m^{-1}$

$m_e = 9.11 \times 10^{-31}\ kg = 511\ keV/c^2 = 5.486 \times 10^{-4}\ u$ $\quad$ neutral ${}^{12}_{6}\mathrm{C}$ atom mass = 12.0000 u

$m_p = 1.673 \times 10^{-27}\ kg = 938.3\ MeV/c^2 = 1.0073\ u$ $\quad 1\ u = 931.5\ MeV$

$m_n = 1.675 \times 10^{-27}\ kg = 939.6\ MeV/c^2 = 1.0087\ u$

$a_0 = 0.0529\ nm$ $\quad E_0 = -13.6\ eV$ $\quad \alpha = 1/137$ $\quad \sigma = 5.670 \times 10^{-8}\ \mathrm{W/m^2K^4}$

Heisenberg: $\quad \Delta p_x\, \Delta x \sim \hbar \qquad \Delta E\, \Delta t \sim \hbar$

Atomic Physics:

Hydrogen Atom: $\quad E = \dfrac{-13.6\ eV}{n^2}$ with $n = 1, 2, 3, \cdots \quad l < n \qquad -l \le m \le +l$

Generalized Balmer Formula: $\quad \dfrac{1}{\lambda} = R_\infty \left(\dfrac{1}{n_f^2} - \dfrac{1}{n_i^2} \right)$

Angular Momentum: $\quad L^2 = l(l+1)\,\hbar^2 \qquad l = 0, 1, 2, 3, ..$ (orbital)

$\quad L_z = \left|\vec{L}\right| \cos\theta = m\,\hbar \qquad -l\ \le\ m_l\ \le\ l$ (integer steps)

Magnetic Moment: $\quad$ orbital: $\vec{\mu} = \dfrac{q}{2\,M}\vec{L}$

for electron: $\quad |\mu_z| = 2 \cdot |m_s|\ \dfrac{e\hbar}{2\,m_e} = \mu_B = 5.8 \times 10^{-5}\ eV/T$

for proton: $\quad |\mu_z| = g_p \cdot |m_s|\ \dfrac{e\hbar}{2\,m_p} = 5.6\ \dfrac{1}{2}\ \mu_N = 8.8 \times 10^{-8}\ eV/T$

Energy of magnetic dipole in B field: $\quad E = -\mu \cdot B$

Energy of particle in 3-dim ∞ square well: $\quad E_{n_x n_y n_z} = \dfrac{(n_x^2 + n_y^2 + n_z^2)h^2}{8mL^2}$

Molecular excitations: $\quad$ vibrational: $E = (n + \dfrac{1}{2})\,\hbar\omega \qquad$ rotational: $E = \dfrac{L^2}{2\,I} = \dfrac{l(l+1)}{2\,I}\,\hbar^2$

Statistical Physics:

Maxwell Boltzmannc distribution: $f_{MB} = A\, e^{-E/kT}$

Bose-Einstein distribution: $f_{BE} = \dfrac{1}{B\, e^{E/kT} - 1}$

Fermi - Dirac distribution: $f_{FD} = \dfrac{1}{e^{(E-E_F)/kT} + 1}$

Boltzmann constant $k = 1.381 \times 10^{-23}\ J/K = 8.617 \times 10^{-5}\ eV/K$

For "gas" of free fermions: $g(E) = \dfrac{8\sqrt{2}\pi m^{3/2}}{h^3}\sqrt{E}$

$$E_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} \qquad\qquad E_m = \frac{3}{5}\, E_F$$

Nuclear Physics:

nuclear radius: $R = (1.2 \times 10^{-15}\, m)\ A^{1/3}$ average binding energy / nucleon $\approx 8\ MeV$

range of interaction: $R = \dfrac{\hbar}{mc} = \dfrac{\hbar c}{mc^2}$

Decay law: $N = N_0\, e^{-\frac{t}{\tau}}$ with $< t >= \tau = \dfrac{1}{\lambda}$ and $T_{1/2} = \dfrac{\ln 2}{\lambda}$

Binding energy B(Z,A) = [Z m_p + N m_n − M_{atom}(Z,A)] c^2

Particle Physics:

Baryon: $Q\,Q\,Q$ Meson: $Q\,\overline{Q}$

Quarks: $\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} \begin{matrix} q = +\frac{2}{3} \\ q = -\frac{1}{3} \end{matrix}$ Leptons: $\begin{pmatrix} e \\ \nu_e \end{pmatrix} \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix} \begin{matrix} q = -1 \\ q = 0 \end{matrix}$

Cosmology:

Luminosity of star: $L = 4\pi r^2 f$ with f = apparent brightness of star

difference in apparent magnitude: $m_1 - m_2 = 2.5 \log(f_1/f_2)$