Momentum and Energy

In classical (Newtonian) mechanics $\vec{p} = m\vec{v}$ and $E = \frac{1}{2}mv^2$, where m is the mass. The equation of motion is given by $\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$.

These expressions must be modified since we know that the Galilean transformation, which $\vec{F} = m\vec{a}$ obeys, is not valid unless $v/c \ll 1$.

Note that for Galilean transformations

$$\vec{p} = m\frac{d\vec{r}}{dt} = \text{scalar mass } \frac{\text{vector displacement}}{\text{scalar time}}.$$

However, time dt does depend on the reference frame and hence is not a scalar ($dt' \neq dt$ in general) in a fully relativistic theory.

We would like to find a scalar under Lorentz Transformations which reduces to dt in the limit when $v/c \ll 1$, or

$$\text{relativistic momentum} = \vec{p} = m\,\frac{\text{vector displacement}}{\text{Lorentz scalar time}}.$$

Note, that for any two events separated by $\Delta x, \Delta y, \Delta z, \Delta t$ the quantity
$c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \equiv (\Delta s)^2$
is a Lorentz invariant or scalar whose value is independent of reference frame.

(You can prove this by exlicit substitutions $\Delta x \to \Delta x'$, etc.)

For infinitesimal differences we obtain $(ds)^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$. Let (dx, dy, dz) represent the change in position of a particle during the time interval dt. Then

$$\begin{aligned}\frac{(ds)^2}{c^2} &= (dt)^2\left[1 - \frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}{c^2}\right] \\ &= (dt)^2\left[1 - \frac{v^2}{c^2}\right].\end{aligned}$$

The quantity $\frac{ds}{c}$ is called "proper" time $d\tau$ and is Lorentz invariant ($d\tau = d\tau'$)

$$\begin{aligned}(d\tau)^2 &= (dt)^2\left[1 - \frac{v^2}{c^2}\right] \\ d\tau &= dt\sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma},\end{aligned}$$

where v, γ refer to the motion of the particle.

Note that when $\frac{v}{c} \to 0 \qquad d\tau \to dt$.

We thus define a "relativistic" momentum:

$$\vec{p} = m\frac{d\vec{r}}{d\tau} = m\frac{d\vec{r}}{dt/\gamma} = m\gamma\frac{d\vec{r}}{dt}$$

or

$$\boxed{\vec{p} = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\vec{v} = \gamma m\vec{v}.}$$

For example $p_x = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dx}{dt}$.

Recall that the Lorentz Transformations for coordinates involve **4** quantities (x, y, z, t). The definition of relativistic momentum is $\vec{p} = m\frac{d\vec{r}}{d\tau}$, where m and $d\tau$ are Lorentz scalars. We thus expect $\vec{p}$ to be the first three components of a "Lorentz 4-vector" and to transform like $d\vec{r}$, where $d\vec{r} = (dx, dy, dz)$. [1]

The only problem is, we do not have the momentum analog to dt

$$\begin{array}{lcl} dx & \to & p_x \\ dy & \to & p_y \\ dz & \to & p_z \\ dt & \to & ? \end{array} \qquad \text{analogy}$$

Let's call the momentum component analogous to dt by the letter $\mathcal{E}$.

What is the letter $\mathcal{E}$ equal to?

$$\text{since} \quad \begin{aligned} p_x &= m\frac{dx}{d\tau} \\ p_y &= m\frac{dy}{d\tau} \\ p_z &= m\frac{dz}{d\tau} \end{aligned}$$

We would expect $\mathcal{E} = m\ \frac{dt}{d\tau} = m\ \frac{dt}{dt/\gamma} = m\ \gamma$
or $\mathcal{E} = m\ \left[1-\frac{v^2}{c^2}\right]^{-1/2}$.

[1] We speak of a "Lorentz 4-vector" $ds = (dx, dy, dz, icdt)$ whose length $(ds)^2 = ds \cdot ds = dx \cdot dx + dy \cdot dy + dz \cdot dz - c^2 dt \cdot dt$ is invariant under Lorentz Transformations ("rotations"). The fourth component must be of the form

$$p_4 = \gamma m\frac{ds_4}{dt} = i\gamma mc \qquad (\text{since} \qquad ds_4 = icdt).$$

Let's do Taylor series expansion in v^2/c^2:

$$f(x) = f(0) + \left.\frac{df}{dx}\right|_{x=0} x + \frac{1}{2}\left.\frac{d^2 f}{dx^2}\right|_{x=0} x^2 + \cdots$$

where $x = v^2/c^2 \quad f(x) = m(1-x)^{-1/2}$

so $\mathcal{E} = m\ \left[1 + \frac{1}{2}\ \frac{v^2}{c^2} + \frac{3}{8}\ \frac{v^4}{c^4} + \cdots\right]$.

If we multiply both sides with c^2

$$c^2\mathcal{E} = \underbrace{mc^2}_{\text{rest-mass energy}} + \underbrace{\frac{1}{2}mv^2}_{\text{looks like kinetic energy}} + \underbrace{\frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right)}_{\to 0 \text{ if } v/c\to 0} + \cdots .$$

Since the rest-mass energy is mc^2, we interpret the fourth component of the momentum 4-vector, $p_4 = i\ \frac{E}{c}$, with $E = c^2\mathcal{E} = \gamma mc^2$ as the relativistic energy of a moving particle of rest mass m.

Note that if $v \ll c \quad E = \gamma mc^2 = \frac{1}{\sqrt{1-v^2/c^2}}\ mc^2 \approx \left(1 + \frac{v^2}{2c^2}\right)\ mc^2 = mc^2 + \frac{1}{2}mv^2$.

With this definition

p_x	transforms like	dx
p_y	"	dy
p_z	"	dz
$\frac{E}{c^2}$	"	dt

or

$$\begin{aligned} p'_x &= \gamma\ \left(p_x - \frac{vE}{c^2}\right) \\ p'_y &= p_y \\ p'_z &= p_z \\ \frac{E}{c^2}' &= \gamma\ \left(\frac{E}{c^2} - \frac{v}{c^2}\ p_x\right) \qquad \text{or} \qquad E' = \gamma\left[E - v\ p_x\right]. \end{aligned}$$

In the same way that $(ds)^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ is Lorentz invariant

$$c^2\left(\frac{E}{c^2}\right)^2 - (p_x)^2 - (p_y)^2 - (p_z)^2 \qquad \text{is } \underline{\text{invariant}}$$

$$= \frac{E^2}{c^2} - p^2 = \frac{\left(\gamma mc^2\right)^2}{c^2} - \left(\gamma mv^2\right)^2 = m^2c^2\underbrace{\left(\gamma^2 - \beta^2\gamma^2\right)}_{1} = \underbrace{m^2c^2}_{\text{invariant}},$$

since the identity $\frac{1}{1-v^2/c^2} - \frac{v^2/c^2}{1-v^2/c^2} = 1$ or $\gamma^2 - \beta^2\gamma^2 = 1$ is a Lorentz invariant, because 1 is a constant. Since the rest-mass m and the speed of light c are both constants, m^2c^2 is also a Lorentz invariant.

We can rewrite this by multiplying by c^2 to obtain

$$\boxed{E^2 - (pc)^2 = (mc^2)^2}$$

This fundamental relationship between energy, momentum, and "rest-mass" is true for all particles.

Recall that $E = \gamma mc^2 = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right) + \cdots$,
which reduces to the expected $\frac{1}{2}mv^2$ when $v/c \to 0$, except for the constant term mc^2. Now, when $v \ll c$, the term $mc^2 \gg \frac{1}{2}mv^2$. The term mc^2 is called the "rest energy" or "rest-mass energy" of a particle. For example, if you take a particle with zero kinetic energy and an antiparticle with zero kinetic energy and let them annihilate, the amount of released energy is $2(mc^2)$.

To complete the relativistic definitions, the relativistic kinetic energy is
$E - mc^2 = mc^2[\gamma - 1]$.

We showed above that $E^2 - (pc)^2$ is invariant, that is $E^2 - (pc)^2 = (E')^2 - (p'c)^2 = (mc^2)^2$ for a single particle with rest-mass m.

This result can be extended to a collection of particles:

$$\begin{aligned} \text{let} \quad E &= E_1 + E_2 + E_3 + \cdots \\ p_x &= p_{1x} + p_{2x} + p_{3x} + \cdots \\ p_y &= p_{1y} + p_{2y} + p_{3y} + \cdots \\ p_z &= p_{1z} + p_{2z} + p_{3z} + \cdots \end{aligned}$$

then $E^2 - (pc)^2 = (E')^2 - (p_xc)^2 - (p_yc)^2 - (p_zc)^2$ is also invariant:

$$E^2 - (pc)^2 = (E')^2 - (p'c)^2 = \left(m_{\text{eff}}\, c^2\right)^2,$$

where $m_{\text{eff}}\, c^2$ is the "effective" mass of the system $\cdots$ and where $m_{\text{eff}}\, c^2$ is the same in all Lorentz frames.

To complete the story, Newton's equation of motion

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt} \qquad \text{becomes} \qquad \vec{F} = \frac{d}{dt}\left(\frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}\,\vec{v}\right).$$