Derivation of Lorentz Transformations

Consider two coordinate systems \((x, y, z, t)\) and \((x', y', z', t')\) that coincide at \(t = t' = 0\). The unprimed system is stationary and the primed system moves to the right along the \(x\)-direction with speed \(v\):

At time \(t = t' = 0\), an isotropic light pulse is generated at \(x = x' = 0, y = y' = 0, z = z' = 0\):

Since the speed of light is the same (= \(c\)) in both systems, the wave front will satisfy both

\[
\begin{align*}
    x^2 + y^2 + z^2 &= (ct)^2 \\
    x'^2 + y'^2 + z'^2 &= (ct')^2.
\end{align*}
\]

If we substitute Galilean transformation \(x' = x - vt; \quad y' = y; \quad z' = z; \quad t' = t\) into Eq. 2, we obtain:

\[
x^2 + y^2 + z^2 \quad -2xvt + v^2 t^2 = (ct)^2.
\]

not compatible with Eq. 1

The Galilean transformation does not work. Note that the unwanted terms above involve both space and time. If we stick with the reasonable assumption that \(y = y'\); \(z = z'\) (implying that there is no effect perpendicular to the relative motion), then the only way we can avoid an unwanted term such as \((-2xvt + v^2 t^2)\) above is to assume that \(t'\) is a function of both \(x\) and \(t\).

The question arises then how \((x, y, z, t)\) and \((x', y', z', t')\) are related. If both coordinate systems are inertial (that is, no relative acceleration), then a particle moving along a straight line in one system must move along a straight line in the other. Otherwise we would introduce spurious forces into the system with a curved trajectory. Hence we require linear transformations.
The most general, linear transformation between \((x, t)\) and \((x', t')\) can be written as:

\[
x' = a_1 x + a_2 t
\]
\[
t' = b_1 x + b_2 t,
\]
where \(a_1, a_2, b_1, b_2\) are constants that can only depend on \(v\), the velocity between the coordinate systems, and on \(c\).

Before substituting Eqs. 3 and 4 into Eq. 2, we note that the origin of the primed frame \((x' = 0)\) is a point that moves with speed \(v\) as seen in the unprimed frame. Its location in the unprimed frame is given by \(x = vt\).

So Eq. 3 must satisfy:

\[
0 = a_1 x + a_2 t, \quad x = -\frac{a_2}{a_1} t = vt \quad \text{or} \quad \frac{a_2}{a_1} = -v.
\]

Let’s now rewrite Eq. 3:

\[
x' = a_1 \left( x + \frac{a_2}{a_1} t \right) = a_1 (x - vt).
\]

Then we have eliminated \(a_2\). Let’s now substitute Eqs. 3 and 4 into Eq. 2, then we obtain

\[
[a_1(x - vt)]^2 + y^2 + z^2 = c^2 [b_1 x + b_2 t]^2.
\]

Expanding and rearranging gives:

\[
(a_1^2 - c^2 b_1^2) x^2 + y^2 + z^2 = (c^2 b_2^2 - a_1^2 v^2) t^2 + (2b_1 b_2 c^2 + 2a_1^2 v) xt.
\]

The only way this is consistent with \(x^2 + y^2 + z^2 = c^2 t^2\) (Eq. 1), is if:

\[
a_1^2 - c^2 b_1^2 = 1, \tag{5}
\]
\[
c^2 b_2^2 - a_1^2 v^2 = c^2, \tag{6}
\]
\[
2b_1 b_2 c^2 + 2a_1^2 v = 0. \tag{7}
\]

We can solve these three equations for the three unknowns \(a_1, b_1, b_2\). A little bit of algebra gives:

from Eqs. 5 and 6, we get

\[
b_1^2 = \frac{a_1^2 - 1}{c^2}
\]
\[
b_2^2 = 1 + \frac{v^2}{c^2} a_1^2,
\]

squaring Eq. 7 and substituting Eqs. 5 and 6 gives:

\[
b_1^2 b_2^2 c^4 = a_1^4 v^2 \rightarrow \left( \frac{a_1^2 - 1}{c^2} \right) \left( 1 + \frac{v^2}{c^2} a_1^2 \right) c^4 = a_1^4 v^2
\]
or

\[(a_1^2 - 1) (c^2 + v^2 a_1^2) = a_1^4 v^2 \rightarrow a_1^2 c^2 - v^2 a_1^2 = 0\]

and \[a_1 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma\]

we chose +\(\sqrt{}\) since as \(v \rightarrow 0\) \(x' = x\), and not \(x' = -x\).

Substituting \(a_1 = \gamma\) into Eqs. 5 and 6, gives

\[\frac{1}{1 - \frac{v^2}{c^2}} - c^2 b_1^2 = 1 \quad \text{or} \quad b_1^2 = \frac{v^2}{c^2} \left(\frac{1}{c^2 - v^2}\right) = \frac{v^2}{c^4} \gamma^2\]

\[c^2 b_2^2 - \frac{1}{1 - \frac{v^2}{c^2}} v^2 = c^2 \quad \text{or} \quad b_2^2 = \frac{1}{1 - \frac{v^2}{c^2}} = \gamma^2\]

We chose \([b_2 = \sqrt{\gamma^2} = +\gamma]\), since \(t' = t\) and not \(t' = -t\) when \(v \rightarrow 0\),

and \([b_1 = -\frac{v}{c^2} \gamma]\), where the minus sign is required by Eq. 7: \(b_1 = -\frac{a_1^2 v}{b_2 c^2} = -\frac{v}{c^2} \gamma\)

So we summarize our new (Lorentz) transformations:

\[
\begin{align*}
  x' &= \gamma (x - vt) \\
  y' &= y \\
  z' &= z \\
  t' &= \gamma \left(t - \frac{v}{c^2} x\right)
\end{align*}
\]

Inverting primed and unprimed coordinates \((v \rightarrow -v)\) gives:

\[
\begin{align*}
  x &= \gamma (x' + vt') \\
  y &= y' \\
  z &= z' \\
  t &= \gamma \left(t' + \frac{v}{c^2} x'\right)
\end{align*}
\]

Note, that when \(v << c\), or \(\frac{v}{c} \rightarrow 0\), then \(\gamma \rightarrow 1\) and \(\frac{v}{c^2} = \frac{1}{c} \frac{v}{c} \rightarrow 0\), and

\[
\begin{align*}
  x' &= x - vt \\
  y' &= y \\
  z' &= z \\
  t' &= t.
\end{align*}
\]

So Galilean transformations are a limiting case of the Lorentz transformations when \(\frac{v}{c} \rightarrow 0\).