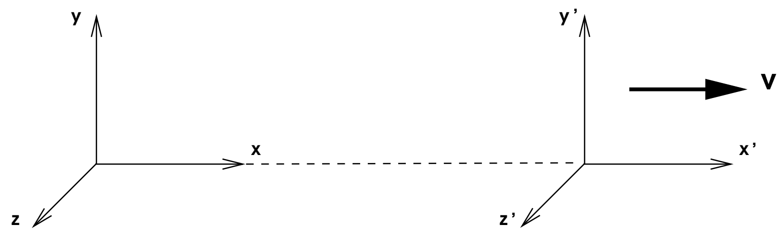
Derivation of Lorentz Transformations

Consider two coordinate systems (x, y, z, t) and (x', y', z', t') that coincide at $t = t' = 0$. The unprimed system is stationary and the primed system moves to the right along the $x-$direction with speed v:



At time $t = t' = 0$, an isotropic light pulse is generated at
$x = x' = 0, y = y' = 0, z = z' = 0$:

Since the speed of light is the same ($= c$) in both systems, the wave front will satisfy both

$$x^2 + y^2 + z^2 = (ct)^2 \tag{1}$$
$$x'^2 + y'^2 + z'^2 = (ct')^2. \tag{2}$$

If we substitute Galilean transformation $\quad x' = x - vt; \quad y' = y; \quad z' = z; \quad t' = t$
into Eq. 2, we obtain:

$$x^2 + y^2 + z^2 \quad \underbrace{-2xvt + v^2t^2}_{\text{not compatible with Eq. 1}} \quad = (ct)^2.$$

The Galilean transformation does not work. Note that the unwanted terms above involve both space and time. If we stick with the reasonable assumption that $y = y'$; $\quad z = z'$ (implying that there is no effect perpendicular to the relative motion), then the only way we can avoid an unwanted term such as $(-2xvt + v^2t^2)$ above is to assume that t' is a function of both x and t.

The questions arises then how (x, y, z, t) and (x', y', z', t') are related. If both coordinate systems are inertial (that is, no relative acceleration), then a particle moving along a straight line in one system must move along a straight line in the other. Otherwise we would introduce spurious forces into the system with a curved trajectory. Hence we require linear transformations.

The most general, linear transformation between (x, t) and $(x' t')$ can be written as:

$$\begin{aligned} x' &= a_1 x + a_2 t && (3) \\ t' &= b_1 x + b_2 t, && (4) \end{aligned}$$

where a_1, a_2, b_1, b_2 are constants that can only depend on v, the velocity between the coordinate systems, and on c.

Before substituting Eqs. 3 and 4 into Eq. 2, we note that the origin of the primed frame $(x' = 0)$ is a point that moves with speed v as seen in the unprimed frame. Its location in the unprimed frame is given by $x = vt$.

So Eq. 3 must satisfy:

$$0 = a_1 x + a_2 t, \qquad x = -\frac{a_2}{a_1} t = vt \quad \text{or} \quad \frac{a_2}{a_1} = -v.$$

Let's now rewrite Eq. 3:

$$x' = a_1 \left(x + \frac{a_2}{a_1} t \right) = a_1 (x - vt).$$

Then we have eliminated a_2. Let's now substitute Eqs. 3 and 4 into Eq. 2, then we obtain

$$[a_1(x - vt)]^2 + y^2 + z^2 = c^2 [b_1 x + b_2 t]^2 .$$

Expanding and rearranging gives:

$$\left(a_1^2 - c^2 b_1^2\right) x^2 + y^2 + z^2 = \left(c^2 b_2^2 - a_1^2 v^2\right) t^2 + \left(2 b_1 b_2 c^2 + 2 a_1^2 v\right) xt.$$

The only way this is consistent with $x^2 + y^2 + z^2 = c^2 t^2$ (Eq. 1), is if:

$$\begin{aligned} a_1^2 - c^2 b_1^2 &= 1, && (5) \\ c^2 b_2^2 - a_1^2 v^2 &= c^2, && (6) \\ 2 b_1 b_2 c^2 + 2 a_1^2 v &= 0, && (7) \end{aligned}$$

We can solve theses three equations for the three unknowns a_1, b_1, b_2. A little bit of algebra gives:

from Eqs. 5 and 6, we get

$$\begin{aligned} b_1^2 &= \frac{a_1^2 - 1}{c^2} \\ b_2^2 &= 1 + \frac{v^2}{c^2} a_1^2, \end{aligned}$$

squaring Eq. 7 and substituting Eqs. 5 and 6 gives:

$$b_1^2 b_2^2 c^4 = a_1^4 v^2 \rightarrow \left(\frac{a_1^2 - 1}{c^2} \right) \left(1 + \frac{v^2}{c^2} a_1^2 \right) c^4 = a_1^4 v^2$$

or

$$\left(a_1^2-1\right)\left(c^2+v^2a_1^2\right)=a_1^4v^2 \rightarrow a_1^2c^2-c^2-v^2a_1^2=0$$

and $\boxed{a_1=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\equiv\gamma}$

we chose $+\sqrt{\ }$ since as $v\rightarrow 0$ $x'=x$, and not $x'=-x$.
Substituting $a_1=\gamma$ into Eqs. 5 and 6, gives

$$\frac{1}{1-\frac{v^2}{c^2}}-c^2b_1^2=1 \quad \text{or} \quad b_1^2=\frac{v^2}{c^2}\left(\frac{1}{c^2-v^2}\right)=\frac{v^2}{c^4}\,\gamma^2$$

$$c^2b_2^2-\frac{1}{1-\frac{v^2}{c^2}}\,v^2=c^2 \quad \text{or} \quad b_2^2=\frac{1}{1-\frac{v^2}{c^2}}=\gamma^2$$

We chose $\boxed{b_2=\sqrt{\gamma^2}=+\gamma}$, since $t'=t$ and not $t'=-t$ when $v\rightarrow 0$,

and $\boxed{b_1=-\frac{v}{c^2}\,\gamma}$, where the minus sign is required by Eq. 7: $b_1=-\frac{a_1^2v}{b_2c^2}=-\frac{v}{c^2}\,\gamma$

So we summarize our new (Lorentz) transformations:

$$\boxed{\begin{aligned} x' &= \gamma\left(x-vt\right)\\ y' &= y\\ z' &= z\\ t' &= \gamma\left(t-\frac{v}{c^2}x\right)\end{aligned}}$$

Inverting primed and unprimed coordinates ($v\rightarrow -v$) gives:

$$\boxed{\begin{aligned} x &= \gamma\left(x'+vt'\right)\\ y &= y'\\ z &= z'\\ t &= \gamma\left(t'+\frac{v}{c^2}x'\right)\end{aligned}}$$

Note, that when $v<<c$, or $\frac{v}{c}\rightarrow 0$, then $\gamma\rightarrow 1$ and $\frac{v}{c^2}=\frac{1}{c}\,\frac{v}{c}\rightarrow 0$, and

$$\begin{aligned} x' &= x-vt\\ y' &= y\\ z' &= z\\ t' &= t.\end{aligned}$$

So Galilean transformations are a limiting case of the Lorentz trnasformations when $\frac{v}{c}\rightarrow 0$.