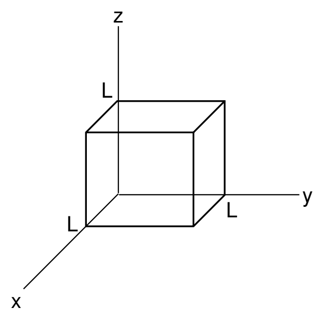
Deriving $\mathbf{n}(\lambda) = \frac{\mathbf{8}\pi}{\lambda^{\mathbf{4}}}$

Consider a cubical cavity with side L, filled with e-m standing waves that have vanishing $\vec{E}$ fields at the cavity walls.



So:

$$\vec{E} = \vec{E}_0 \, \sin(K_x x + K_y y + K_z z)$$

with

$$\begin{aligned} K_x \, L &= m_x \, \pi \\ K_y \, L &= m_y \, \pi \\ K_z \, L &= m_z \, \pi \end{aligned} \qquad \begin{aligned} m_x &= 0, 1, 2, ... \\ m_y &= 0, 1, 2, ... \\ m_z &= 0, 1, 2, ... \end{aligned}$$

The wave vector $\vec{K} = (K_x, K_y, K_z)$ is related to the wavelength λ by:

$$K = |\vec{K}| = \sqrt{K_x^2 + K_y^2 + K_z^2} = \frac{2\pi}{\lambda} = \frac{\pi}{L} \sqrt{m_x^2 + m_y^2 + m_z^2} \equiv \frac{\pi}{L} \, m$$

or

$$m = \frac{2L}{\lambda}$$

Each wave can be represented by a point in a 3-dimensional (m_x, m_y, m_z) space. Consider the number of points in a spherical volume bounded by some maximum m value. (Since $m_x > 0, m_y > 0, m_z > 0$, we are actually dealing with 1/8 of the volume of the sphere).

$$n(m) \equiv \frac{\text{number of states}}{\text{physical volume}} = \frac{N}{L^3} = \frac{1}{8} \left(\frac{4}{3} \pi \, m^3 \right) \frac{2}{L^3} = \frac{\pi m^3}{3L^3}$$

where the extra factor of 2 results from two possible polarizations.

Now,

$$\frac{dn}{dm} \, dm = \frac{dn}{d\lambda} \, |d\lambda| \equiv n(\lambda) \, |d\lambda|$$

so

$$n(\lambda) = \frac{dn}{dm} \left| \frac{dm}{d\lambda} \right| = \frac{\pi m^2}{L^3} \, \frac{2L}{\lambda^2} = \frac{\pi}{L^3} \, \frac{4L^2}{\lambda^2} \, \frac{2L}{\lambda^2} = \frac{8\pi}{\lambda^4}$$

or

$$\underline{n(\lambda) = \frac{8\pi}{\lambda^4}}$$