Physics 390 Fall 2004: Exam #1 Practice Solutions

These are a few problems comparable to those you will see on the exam. They were picked from previous exams. I will provide a sheet with useful constants and equations for the exam.

1: The energy reaching the Earth from the Sun at the top of the atmosphere is described by the 'Solar Constant': 1360 W/m^2. The radius of the Earth is $6.4x10^6$m. Assume that the Earth radiates like a blackbody at a uniform temperature.

a) What value would you estimate for the equilibrium temperature of the Earth?
b) What would be the peak wavelength for thermal emission from the Earth?

a) The Earth is both absorbing solar radiation and emitting its own thermal radiation. These two processes will balance at the equilibrium temperature:
Absorbed solar light: The sun energy arriving is 1360 W/m^2. The area of the Earth which faces the sun is given by $\pi r^2 = 1.3x10^{14}$ m^2. It's *not* half the surface area of the Earth, because the Sun doesn't shine down on every point. This means that the total energy arriving at the top of the atmosphere is about:

$$1360 \, W/m^2 * 1.3x10^{14} m^2 = 1.8x10^{17} \, W$$

Emitted thermal radiation: Assuming the entire Earth is at a uniform temperature, it emits a total power of Area * $\sigma T^4 = 4\pi r^2 * \sigma T^4 = 2.9x10^7$ $W/K^4 * T^4$
Equating these two yields : $T^4 = 1.8x10^{17}$ W / $2.9x10^7$ $W/K^4 = 6.2x10^9$ K^4 or an equilibrium temperature of T = 280 K.

b) $\lambda_{max}T = 2.9x10^{-3}$ mK or $\lambda_{max} = 1.0x10^{-5}$m, or 10μm.

2: An electron is trapped in an infinitely deep one-dimensional potential well. The width of the well is 10^{-9}m.

a) Write an expression for the solutions to the Schrodinger equation for this potential. Your solutions need not be normalized, but they should meet the boundary conditions appropriate for this potential well.
b) Draw the wave function for the n=5 state.
c) What is the difference in energy between the n=4 and the n=5 state?

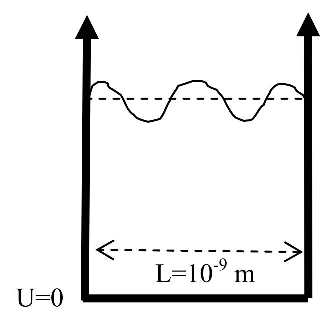
a) The wave functions are the usual free wave solutions, but they must go to zero at x=0 and x=L, so they are:

$$\psi(x)=\sin(n\pi x/L)$$

These waves have energy $E_n = \hbar^2\pi^2 n^2/2mL^2 = n^2*(0.37 \text{ eV})$

b) The n=5 state has five peaks in it, and is shown on the drawing to the right.

c) The energy difference is $E_5 - E_4 = (25 - 16)*0.37$ eV = 3.36 eV



3: In a repeat of the Davisson and Germer electron diffraction experiment a beam of electrons with kinetic energy of 54 eV are fired at a clean surface of Nickel.

a) What is the wavelength of these electrons?
b) If the Ni atoms are arranged in a regular cubic lattice with a spacing of 0.45 nm, what is the largest angle at which a strong signal of scattered electrons will be seen?

a) The wavelength is given by $\lambda = h / p = 1.67\text{x}10^{-10}$ m

b) The scattering relation for electrons off a surface is $n\lambda = D\sin\theta$ or

$$\sin\theta = n\lambda / D = n(1.67\text{x}10^{-10} / 4.5\text{x}10^{-10}) = n{*}0.371$$

For this, the largest allowed value of n is 2, and then we have

$$\theta = \sin^{-1}(0.742) = 47.9°$$

4: X-rays tubes used in dentist's offices often have an accelerating voltage of 80 kV.

a) What is the minimum wavelength such an x-ray tube can produce?
b) What is the maximum wavelength such an x-ray can have after Compton scattering off an electron inside your tooth?
c) Estimate the maximum wavelength such an x-ray can have after scattering off a calcium nucleus in your tooth. A calcium nucleus contains 20 protons and 20 neutrons.

a) An electron accelerated through 80 kV and slammed into a target can create, by Bremmstrahlung radiation, a photon with an energy of 80 keV. This is a wavelength $\lambda = hc/E = 1.55\text{x}10^{-11}$ m

b) In Compton scattering the photon gives up some of its energy to an electron. It will give up the largest possible part of its energy when it scatters directly back along the path on which it entered. Shifts in wavelength due to Compton scattering are given by:

$$\lambda' - \lambda = (h/m_e c)(1 - \cos\theta)$$

When the photon scatters straight back, $\theta=180°$, and $\cos\theta = -1$, so

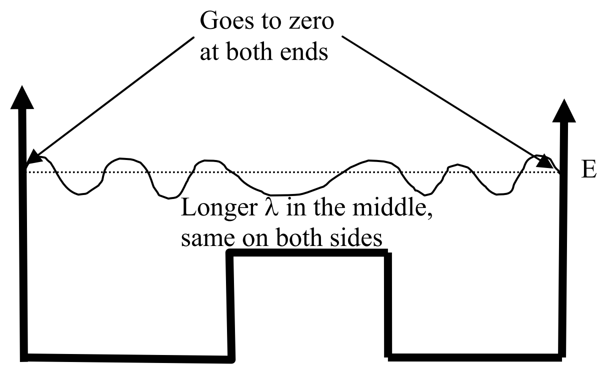
$$\lambda' - \lambda = 2(h/m_e c) \qquad \text{or} \qquad \lambda' = \lambda + 2(h/m_e c)$$

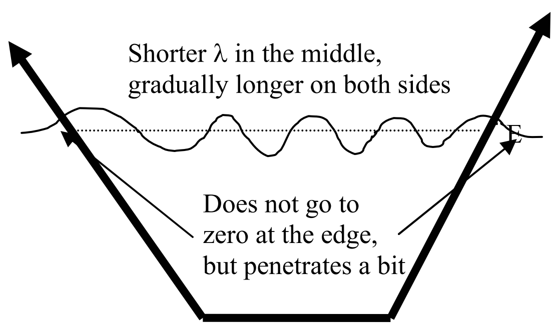
where this factor $(h/m_e c) = 2.4\text{x}10^{-12}$ m is called the Compton wavelength of the electron. This tells us:

$$\lambda'_{max} = \lambda + 4.8\text{x}10^{-12}\text{ m} = 2.03\text{x}10^{-11}\text{ m}$$

c) Scattering off the calcium nucleus is just like scattering off an electron *except* that the calcium nucleus is roughly 40*1800 = 72,000x as heavy as the electron. This means the Compton wavelength of this nucleus is 72,000 times as small as that of the electron. Shifts in the wavelength of backscattered photons are similarly reduced. Since the shifts are so tiny, the incoming photon scatters back with just about its original energy.

5: Draw qualitative wavefunctions which represent solutions to the Schroedinger equation for particles with the energies shown confined by the following potentials. Please note with words any particular features you wish to stress.





6: One solution for a particle in a one dimensional infinite square well of width L which extends from x=0 to x=L is:

$$\psi_2(x) = [\sqrt{(2/L)}]\sin(2\pi x/L)$$

This is a properly normalized wave function. That is $\int \psi_2^* \psi_2 dx = 1$. The operator for the momentum is $O_p = (\hbar/i)\partial/\partial x$.

a) Calculate the expectation value for the momentum $\langle p \rangle$
b) Calculate the expectation value for the momentum squared $\langle p^2 \rangle$
c) What is the uncertainty in the momentum Δp for this state?
d) Show that this is roughly the value you would expect from the Heisenberg uncertainty principle.

In answering these you may find the following relation useful:

$$\sin A\cos B = \tfrac{1}{2}[\sin(A+B) + \sin(A-B)]$$

a) $\langle p \rangle = \int \psi^*(\hbar/i)\partial/\partial x\psi dx = (2/L) \int \sin(2\pi x/L)(\hbar/i)(2\pi/L)\cos(2\pi x/L)dx$

Note, in this case the wavefunction is non-zero *only* from 0 to L, so the integration is taken over only this range. The above can be rewritten:

$$\langle p \rangle = \int \psi^*(\hbar/i)\partial/\partial x\psi dx = (2\pi\hbar/iL^2) \int^L_0 \sin(2\pi x/L)\cos(2\pi x/L)dx$$

This is the integral of an even function (the sin) times an odd function (the cos) so the integral is zero. There are several other ways you could see this. You could use the substitution given above and work this out, or you could just recognize that for a particle trapped in a box the average momentum *must* be zero! It can't, on average, be moving anywhere, because it's trapped in a box!

b) $\langle p^2 \rangle = \int \psi^*(-\hbar^2)\partial^2/\partial x^2\psi dx = (2/L) \int \sin(2\pi x/L)(-\hbar^2)\partial^2/\partial x^2\sin(2\pi x/L)dx = (8\pi^2\hbar^2/L^3) \int_0^L \sin^2(2\pi x/L)dx$

Here the integral of the $\sin^2$ over any number of half wavelengths is equal to ½ times the range over which you integrate. In this case that's L/2. This fact is one you should eventually know, that the average value of $\sin^2$ or $\cos^2$ is ½. It comes into knowing that the RMS value of something which is varying sinusoidally is ½ the maximum value for example. This gives:

$$\langle p^2 \rangle = 4\pi^2\hbar^2/L^2$$

c) The momentum uncertainty can be estimated:

$$\sigma_p = \sqrt{(\langle p^2 \rangle - \langle p \rangle^2)} = 2\pi\hbar/L$$

d) Combining this with the uncertainty in x (which is about L), we find

$$\Delta x\Delta p = L * (2\pi\hbar/L) = 2\pi\hbar$$

This is certainly allowed by the uncertainty principle. What would we have to do to make a state with *minimum* uncertainty? We would have to combine several states of fixed energy (the energy eigenfunctions) to build up a wave packet with minimum x and p uncertainty.