Solving the Quantum Harmonic Oscillator Problem

Schrödinger's equation for the harmonic oscillator potential is given by:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{2}Kx^2\Psi = i\hbar\frac{\partial\Psi}{\partial t}. \tag{1}$$

For stationary, bound-state solutions $\Psi(x,t) = \psi(x)e^{-i\frac{E}{\hbar}t}$ and the x-dependent part satisfies:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}Kx^2\psi(x) = E\psi(x). \tag{2}$$

It is not obvious how to solve the above equation to find the allowed values of E and the corrresponding wavefunction $\psi(x)$. In fact, there are some general techniques for solving differential equations. However, this problem can be solved (exactly !) using a beautiful trick invented by Schrödinger:

Let's define $\omega = \sqrt{\frac{K}{m}}$ and $y = \sqrt{\frac{m\omega}{\hbar}}\,x$ or $dy = \sqrt{\frac{m\omega}{\hbar}}\,dx$. Note, that ω is the classical oscillator speed: $x = x_0\cos\omega t$, which satisfies: $m\frac{d^2x}{dt^2} = -Kx$. Therefore, substituting $x = \sqrt{\frac{\hbar}{m\omega}}\,y$ and $dx = \sqrt{\frac{\hbar}{m\omega}}\,dy$ into the above Schrödinger's equation (Eq. 2), we get:

$$-\frac{\hbar^2}{2m}\frac{1}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\frac{d^2\psi}{dy^2} + \frac{1}{2}(m\omega^2)\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2 y^2\psi = E\psi$$

and

$$\frac{d^2\psi}{dy^2} - y^2\psi = -\frac{2E}{\hbar\omega}\psi \qquad \text{or} \qquad \left[\frac{d^2}{dy^2} - y^2\right]\psi = -\frac{2E}{\hbar\omega}\psi. \tag{3}$$

This can be written as

$$\left[\left(\frac{d}{dy} - y\right)\left(\frac{d}{dy} + y\right) - 1\right]\psi = -\frac{2E}{\hbar\omega}\psi. \tag{4}$$

To see this:

$$\begin{aligned}\left(\frac{d}{dy} - y\right)\left(\frac{d}{dy} + y\right)\psi - \psi &= \left(\frac{d}{dy} - y\right)\left(\frac{d\psi}{dy} + y\psi\right) - \psi \\ = \quad \frac{d^2\psi}{dy^2} - y\frac{d\psi}{dy} + y\frac{d\psi}{dy} + \psi - y^2\psi - \psi &= \frac{d^2\psi}{dy^2} - y^2\psi.\end{aligned}$$

So Schrödinger's equation for the harmonic oscillator becomes:

$$\left(\frac{d}{dy} - y\right)\left(\frac{d}{dy} + y\right)\psi = \left(1 - \frac{2E}{\hbar\omega}\right)\psi. \tag{5}$$

Let's now "operate" on this equation from the left side with $\left(\frac{d}{dy}+y\right)$. Then we obtain:

$$\left(\frac{d}{dy}+y\right)\left(\frac{d}{dy}-y\right)\left(\frac{d}{dy}+y\right)\psi=\left(1-\frac{2E}{\hbar\omega}\right)\left(\frac{d}{dy}+y\right)\psi.$$

But

$$\begin{aligned} &\left(\frac{d}{dy}+y\right)\left(\frac{d}{dy}-y\right)f=\left(\frac{d}{dy}+y\right)\left(\frac{df}{dy}-yf\right) \\ =\quad &\frac{d^2f}{dy^2}+y\frac{df}{dy}-y\frac{df}{dy}-f-y^2f=\left(\frac{d^2}{dy^2}-y^2-1\right)f. \end{aligned}$$

This is true for any function of y, for example $f=\left(\frac{d}{dy}+y\right)\psi$. Therefore,

$$\left(\frac{d^2}{dy^2}-y^2\right)\left(\frac{d}{dy}+y\right)\psi-\left(\frac{d}{dy}+y\right)\psi=\left(1-\frac{2E}{\hbar\omega}\right)\left(\frac{d}{dy}+y\right)\psi.$$

Rearranging gives:

$$\left(\frac{d^2}{dy^2}-y^2\right)\left[\left(\frac{d}{dy}+y\right)\psi\right]=\left(2-\frac{2E}{\hbar\omega}\right)\left[\left(\frac{d}{dy}+y\right)\psi\right].$$

or

$$\left(\frac{d^2}{dy^2}-y^2\right)\left[\left(\frac{d}{dy}+y\right)\psi\right]=-\frac{2(E-\hbar\omega)}{\hbar\omega}\left[\left(\frac{d}{dy}+y\right)\psi\right]. \tag{6}$$

But recall Eq. 3

$$\left(\frac{d^2}{dy^2}-y^2\right)\psi=-\frac{2E}{\hbar\omega}\psi.$$

These equations have the same form if we define $\psi'=\left(\frac{d}{dy}+y\right)\psi$ and $E'=E-\hbar\omega$. Because then:

$$\left(\frac{d^2}{dy^2}-y^2\right)\psi'=-\frac{2E'}{\hbar\omega}\psi'. \tag{7}$$

What this means is that **if** we have found a solution $\psi(y)$ corresponding to energy E, **then** $\left(\frac{d}{dy}+y\right)\psi=\frac{d\psi}{dy}+y\psi$ will **also** be a solution, and its corresponding energy will be $(E-\hbar\omega)$.

We can just keep going like this. Each time the energy is lowered by $\hbar\omega$. This means that the spacing of the energy levels of the quantum harmonic oscillator is $\hbar\omega$.