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Repeated games with present-biased preferences

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Abstract

We study infinitely repeated games with perfect monitoring, where players have β–δ preferences. We compute the continuation payoff set using recursive techniques and then characterize equilibrium payoffs. We then explore the cost of the present-time bias, producing comparative statics. Unless the minimax outcome is a Nash equilibrium of the stage game, the equilibrium payoff set is not monotonic in β or δ. Finally, we show how the equilibrium payoff set is contained in that of a repeated game with smaller discount factor.

JEL classification: C73

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1. Introduction

A decision-maker without geometric time preferences has a dynamic inconsistency problem: How should he behave if he knows that his future behavior might well undo his best laid current plans? If he cannot precommit his future behavior, then the best he can do, according to Strotz [16], is to resign himself to the intertemporal conflict, choosing "the best plan among those he will actually follow." Namely, after any history of choices, he takes an action that maximizes his utility, given his future utility maximizing strategies. Strotz–Pollak equilibrium, as Peleg and Yaari [13] formalize it, is then a strategy immune to one-shot deviations, accounting for the intertemporal conflict. It has since also become conventional to interpret this as a subgame perfect equilibrium

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(SPE) amongst temporally distinct selves: "a specification of a strategy for each player such that agent t's choice after any history of play is optimal given all other agents' strategies" [8].

The literature on present-biased (or 'hyperbolic') discounting has focused largely on decision makers acting either alone or in perfectly competitive environments. Examples include the savings consumption decision [10,12], the smoking decision [7], the labor supply decision [5], and optimal taxation [9]. This paper instead analyzes the strategic behavior of many hyperbolic discounters interacting in a non-cooperative game. [1] Specifically, we study infinitely repeated games with perfect monitoring, where players have β–δ preferences—i.e. with the quasi-geometric discount sequence $1, \beta\delta, \beta\delta^2, \ldots$.

Our contributions are threefold: First, we develop a tractable machinery to characterize the Strotz–Pollak equilibrium payoff set. Second, we relate Strotz–Pollak equilibrium to subgame perfection. And finally, we flesh out some non-standard comparative statics, and use them to bound the payoff set.

We first provide a two-step procedure to characterize the set of Strotz–Pollak equilibrium payoffs: We characterize the continuation payoff set by a recursive methodology adapted from Abreu et al. [2,3] (hereafter, APS), using different weights on current versus future payoffs than found in the incentive constraints. From this recursive set, the equilibrium payoffs are computed.

Subgame perfection is the natural touchstone of credibility in repeated games. But while Strotz–Pollak equilibrium is a weaker notion, it coincides with subgame perfection under geometric payoff discounting. But outside this time-consistent world, the one-shot deviation principle fails, and these solution concepts diverge. Still, we show that the generated action paths and payoffs coincide. So unlike the single-agent case, one can always support a Strotz–Pollak equilibrium so that different "incarnations" of a player agree on the best strategy, given the other players' strategies. No intertemporal conflict ultimately arises among the multiple selves.

To ensure convex sets of payoffs, we then add a public randomization device. In a key difference from geometric discounting, the continuation payoffs are now a possibly proper subset of the equilibrium payoffs. This wedge drives some interesting comparative statics, and allows us to calibrate the cost of the present-time bias. We show that variation in $\beta < 1$ matters more than that of the long-term discount factor $\delta < 1$. To see this, suppose that an eternal one util flow discounted by the constant factor $\Delta < \delta$ has the same present value as it does with β–δ preferences. Fixing Δ, the continuation and equilibrium payoff set shrinks if β falls and δ rises—despite the greater weight δ on continuation payoffs in the recursion. For this reason, this "compensated comparative static" cannot possibly follow from a proof like APS employ, and we instead develop a novel proof by half space intersection.

The sole comparative static in APS is that the equilibrium payoff set expands in the discount factor. Surprisingly, this folk result now fails, insofar as the payoff set is not separately monotonic in β and δ. Our counterexample exploits the wedge between the worst equilibrium and continuation payoffs. But when the minimax point of the stage game is a Nash equilibrium, like the Prisoner's Dilemma, the worst equilibrium and continuation payoffs coincide, and the intuitive result is restored.

We finally seek to shed light on the size of the equilibrium payoff set for fixed β, δ. We sandwich it between the payoff sets of a standard repeated game with different discount factors. Our comparative static bounds it above using the discount factor Δ. But to really understand how much the present bias hurts payoffs, we seek a lower bound. The factor $\beta\delta$ clearly discounts all future payoffs less than the β–δ discounter does, and is a strong candidate for a lower bound.

[1] In independent work, Sarafidis [15] focuses on naive players engaged in non-repeated games.

While valid for a class of symmetric prisoner's dilemma games, we show by example that it generally fails.

The next section outlines the model and solution concepts. Section 3 develops the recursive characterization results, and relates Strotz–Pollak and SPE. We explore the effects of the present bias on the payoffs in Section 4, developing some comparative statics. The paper proceeds by illustrative examples. Technical proofs are found in the appendix.

2. The model

We analyze infinitely repeated games with perfect monitoring. The only departure from the standard such model is that we assume quasi-geometric (or β–δ) discounting. Such preferences were first studied in economics by Phelps and Pollak [14], and are sometimes called hyperbolic or quasi-hyperbolic (e.g. see [10]).

Denote the stage game by $G = (N, (A_i)_{i\in N}, (\pi_i)_{i\in N})$, where $N = \{1, \ldots, n\}$ is a finite set of players, $A_i = \{1, \ldots, I_i\}$ is player i's finite set of actions, and π_i is player i's payoff function from $A = \times_{i\in N} A_i$ to the real line $\mathbb{R}$. We shall assume that the stage game has a Nash equilibrium in pure strategies. Let V denote the set of feasible and weakly individually rational payoffs of G.

The repeated game $G^\infty(\beta, \delta)$ begins at stage 0, with the null history $h^0 = \emptyset$. At the beginning of stage k, each player observes the *history* h^k of actions chosen at all previous stages. If the action profile $a^k = (a_1^k, \ldots, a_n^k)$ is chosen at stage $k \in \{0, 1, \ldots\}$, then player i's stage-t ($t = 0, 1, \ldots$) discounted total payoff is

$$\pi_i(a^t) + \beta \sum_{k=1}^{\infty} \delta^k \pi_i(a^{t+k}), \tag{1}$$

where β and δ (both between 0 and 1) are discount factors common to all players. The parameter β captures a player's bias for the present, for it implies that she assigns more weight to the stage-k versus stage $k+1$ payoffs at stage k than she did before stage k. This present bias is the source of dynamic inconsistency in the model, since current and past preferences over 'today' versus 'tomorrow' differ.

Given discount factors β and δ, define the *effective discount factor* $\Delta(\beta, \delta)$ by

$$1 + \beta\delta + \beta\delta^2 + \cdots = 1 + \Delta(\beta, \delta) + \Delta(\beta, \delta)^2 + \cdots \quad \Rightarrow \quad \Delta(\beta, \delta) = \frac{\beta\delta}{1 - \delta + \beta\delta}.$$

Critical throughout this paper is that $\Delta(\beta, \delta) < \delta$. Player i's stage-t discounted *average* payoff equals his stage-t discounted payoff scaled by $(1 - \Delta(\beta, \delta))$.

Let A^k denote the k-fold Cartesian product of A, and H^k the set of all stage-k action histories. Obviously, $H^k = A^k$. A *pure strategy* for player i is a map f_i from $H = \bigcup_{k=0}^{\infty} H^k$ to A_i. Since we restrict attention to pure strategies (see [1]), we henceforth and throughout the paper drop the qualifier "pure".

Denote by $f_i^{|h}$ player i's strategy induced by f_i in the *subgame* after history $h \in H$. By definition,

$$f_i^{|h}(h') = f_i(\{h, h'\})$$

for all $i \in N$, all $a_i \in A_i$, and all conjoined histories $\{h, h'\} \in H$, listing all actions in h followed by those in h'. As usual, we write $f^{|h} = (f_1^{|h}, \ldots, f_n^{|h})$.

Given a strategy profile $f = (f_1, \ldots, f_n)$ and a history $h \in H$, the discounted total payoff (1) of player i can be rewritten in discounted average form

$$u_i(f^{|h}|\beta, \delta) = (1 - \Delta(\beta, \delta))\pi_i(f(h)) + \Delta(\beta, \delta)c_i(f^{|\{h, f(h)\}}|\delta),$$

where $c_i(f^{|\{h, f(h)\}}|\delta)$ is his naturally defined *continuation payoff*.

Our solution concept is a simple extension of that used in the decision-theoretic literature on present-biased preferences: A strategy profile $f = (f_1, \ldots, f_n)$ is a *Strotz–Pollak equilibrium* if, for any history $h \in H$, player $i \in N$, and action $a_i \in A_i$,

$$\begin{aligned}&(1 - \Delta(\beta, \delta))\pi_i(f(h)) + \Delta(\beta, \delta)c_i(f^{|\{h, f(h)\}}|\delta)\\ &\quad\geqslant (1 - \Delta(\beta, \delta))\pi_i(a_i', f_{-i}(h)) + \Delta(\beta, \delta)c_i(f^{|\{h, (a_i', f_{-i}(h))\}}|\delta).\end{aligned}$$

Hence, a strategy profile is a Strotz–Pollak equilibrium if it admits no profitable one-stage deviations. Intuitively, each player then finds it optimal, given his preferences at any stage, to carry out the actions his strategy specified for that stage.

Conditional on the players choosing their actions according to f, the stage-$(k-1)$ continuation payoff can be represented *along the equilibrium path* as follows:

$$c_i(f^{|h^k}|\delta) = (1 - \delta)\pi_i(f(h^k)) + \delta c_i(f^{|\{h^k, f(h^k)\}}|\delta).$$

As we later see, the recursive structure of the continuation payoffs allows us to adapt the APS technique to compute the Strotz–Pollak equilibrium continuation payoff set.

For the game $G^\infty(\beta, \delta)$, let $\Sigma^{S\&P}(\beta, \delta)$ denote the set of Strotz–Pollak equilibrium strategy profiles, $U(\beta, \delta)$ the set of Strotz–Pollak equilibrium payoffs, and $C(\beta, \delta)$ the set of Strotz–Pollak equilibrium continuation payoffs. Thus,

$$U(\beta, \delta) = \bigcup_{f \in \Sigma^{S\&P}(\beta,\delta)} u(f|\beta, \delta) \quad \text{and} \quad C(\beta, \delta) = \bigcup_{f \in \Sigma^{S\&P}(\beta,\delta)} c(f|\delta),$$

where $u(f|\beta, \delta) = (u_1(f|\beta, \delta), \ldots, u_n(f|\beta, \delta))$ and $c(f|\delta) = (c_1(f|\delta), \ldots, c_n(f|\delta))$.

3. Characterization results

3.1. Recursive structure and construction of $U(\beta, \delta)$

We now turn to a constructive characterization of the set of Strotz–Pollak equilibrium payoffs, adapting the method of APS. Our two-step procedure can be succinctly explained as follows. First, we exploit the recursive structure of continuation payoffs, and thereby characterize the set $C(\beta, \delta)$ as the largest fixed point of an operator. Next, we construct the set $U(\beta, \delta)$ using continuation payoffs drawn from $C(\beta, \delta)$.

As in APS, fix a discount factor $\Delta \in (0, 1)$ and payoff set $W \subset \mathbb{R}^n$. For any action profile $a \in A$ and function κ from A to continuation values W, we say that (a, κ) is Δ-*admissible* w.r.t. W if, for all players $i \in N$, and actions $a_i' \in A_i$,

$$(1 - \Delta)\pi_i(a_i, a_{-i}) + \Delta\kappa_i(a_i, a_{-i}) \geqslant (1 - \Delta)\pi_i(a_i', a_{-i}) + \Delta\kappa_i(a_i', a_{-i}).$$

As is standard, this means that the current and continuation payoffs $\pi_i(a)$ and $\kappa_i(a)$ can be enforced in an incentive compatible fashion, in which any deviation by any player i to other actions a_i' can be punished by continuation payoffs $\kappa_i(a_i', a_{-i}) \in W$.

A set $W \subset V$ is called *self-generating* if $W \subset B(W|\beta, \delta)$, for the operator

$$B(W|\beta, \delta) = \{v \in V : v = (1-\delta)\pi(a) + \delta\kappa(a) | (a, \kappa) \text{ is } \Delta(\beta, \delta)\text{-admissible w.r.t. } W\}.$$

For standard repeated games with geometric discounting ($\beta = 1$), APS have shown that the self-generating sets are candidate equilibrium payoff sets. For continuation payoffs W are themselves enforceable as equilibrium payoffs whose continuations are drawn from W. Just like the decision theory setting (see [14]), the recursive structure applies only to the set of equilibrium *continuation* payoffs.

Theorem 1 (*Characterization*). *Fix* $0 < \beta \leqslant 1$ *and* $0 < \delta < 1$.

(a) *If* $W \subset \mathbb{R}^n$ *is a bounded self-generating set, then* $B(W|\beta, \delta) \subset C(\beta, \delta)$.
(b) Factorization: $C(\beta, \delta)$ *is the largest fixed point of the operator* $B(\cdot|\beta, \delta)$:

$$C(\beta, \delta) = \{(1-\delta)\pi(a) + \delta\kappa(a) | (a, \kappa) \textit{ is } \Delta(\beta, \delta)\textit{-admissible w.r.t. } C(\beta, \delta)\}.$$

Further, given $C(\beta, \delta)$, *the equilibrium payoff set* $U(\beta, \delta) \subset \mathbb{R}^n$ *satisfies*:

$$U(\beta, \delta) = \{v = (1 - \Delta(\beta, \delta))\pi(a) + \Delta(\beta, \delta)\kappa(a) | (a, \kappa) \textit{ is } \Delta(\beta, \delta)\textit{-admissible w.r.t. } C(\beta, \delta)\}.$$

(c) *The sets* $C(\beta, \delta)$ *and* $U(\beta, \delta)$ *are compact*.

The proofs of (a), (b), and the compactness in part (c) for the set $C(\beta, \delta)$ are gentle adaptations of Theorems 1, 2, and 4 of APS, respectively. To see that $U(\beta, \delta)$ is compact, notice that $U(\beta, \delta) = B(C(\beta, \delta)|1, \Delta(\beta, \delta))$, and the latter is compact, as the operator B preserves compactness (also easy given Lemma 1 in APS).

3.2. Strotz–Pollak equilibrium and subgame perfection

In a Strotz–Pollak equilibrium, each player's strategy is a consistent plan that takes into account his intrinsic present bias—on and off the equilibrium path. In other words, each player's current "self" finds it optimal in every period to choose the actions specified by the strategy profile.

It is well known that in infinitely repeated games with perfect monitoring and geometric discounting, the one-stage deviation principle guarantees that the sets of Strotz–Pollak and subgame perfect equilibria coincide; hence, their equilibrium payoff sets coincide as well. We now analyze this "equivalence" result when $\beta < 1$.

To this end, we call a strategy profile f a *sincere Nash equilibrium* of $G^\infty(\beta, \delta)$ if $u_i(f|\beta, \delta) \geqslant u_i(g_i, f_{-i}|\beta, \delta)$ for all $g_i : H \to A_i$ and all $i \in N$. We call it a *sincere* SPE if $f^{|h}$ is a sincere Nash equilibrium of $G^\infty(\beta, \delta)$ for any history $h \in H$. The qualifier "sincere" is added to emphasize that, in each subgame, a player evaluates the optimality of his strategy using his current preferences. This distinction is clearly immaterial if $\beta = 1$.

While every sincere SPE is Strotz–Pollak, because the latter only checks for one-stage deviations at the beginning of each subgame, the converse fails when $\beta < 1$.

Example 1 (*Strotz–Pollak equilibrium need not be subgame perfect*). Consider an infinitely repeated prisoners' dilemma game with $\beta = \frac{1}{3}$ and $\delta = \frac{6}{7}$, so that $\Delta(\frac{1}{3}, \frac{6}{7}) = \frac{2}{3}$. The stage game is in Fig. 1.

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

Fig. 1. Prisoners' dilemma for Example 1.

It is easy to show that the set $W = \{(0, 0), (1, 1), (\frac{8}{7}, \frac{5}{7}), (\frac{41}{49}, \frac{44}{49})\}$ is self-generating and hence belongs to $C(\frac{1}{3}, \frac{6}{7})$. The vector $(0, 0)$ can be supported by the action profile (D, D) and the continuation vector $(\kappa(C, C), \kappa(C, D), \kappa(D, C), \kappa(D, D)) = ((0, 0), (0, 0), (0, 0), (0, 0))$; this obviously constitutes an admissible pair, because (D, D) is a Nash equilibrium of the stage game. Likewise,

- $(1, 1)$ can be supported by Nash reversion: $((C, C), ((1, 1), (0, 0), (0, 0), (0, 0)))$;
- $(\frac{8}{7}, \frac{5}{7})$ can then be supported by $((D, C), ((0, 0), (0, 0), (1, 1), (0, 0)))$;
- $(\frac{41}{49}, \frac{44}{49})$ can then be supported by $((C, D), ((1, 1), (\frac{8}{7}, \frac{5}{7}), (0, 0), (0, 0)))$.

In each case, one can readily check that the suggested pair is admissible.

We now use the set W of continuation payoffs to support the following Strotz–Pollak equilibrium path: $(C, D), (C, D), (D, C), (C, C), (C, C), \ldots$. This path yields the average discounted payoff vector $u(f\,|\,\frac{1}{3}, \frac{6}{7}) = (\frac{11}{49}, \frac{62}{49})$ via the strategy profile f:

- Stage 0: the pair is $((C, D), ((0, 0), (\frac{41}{49}, \frac{44}{49}), (0, 0), (0, 0)))$;
- Stage 1: after $\{(C, D)\}$, the pair is $((C, D), ((1, 1), (\frac{8}{7}, \frac{5}{7}), (0, 0), (0, 0)))$;
- Stage 2: after $\{(C, D), (C, D)\}$, $((D, C), ((0, 0), (0, 0), (1, 1), (0, 0)))$;
- Stage 3: after $\{(C, D), (C, D), (D, C)\}$, $((C, C), ((1, 1), (0, 0), (0, 0), (0, 0)))\ldots$.

For instance, if a player deviates from the action profile (C, D) at stage 0, and, as a result, the stage-1 public history differs from $\{(C, D)\}$, then the players would play (D, D) forever thereafter. A similar interpretation applies to subsequent stages.

It is not difficult to verify that (i) $u(f^{|\{(C,D)\}}|\frac{1}{3}, \frac{6}{7}) = (\frac{3}{7}, \frac{8}{7})$ and that (ii) this strategy profile is a Strotz–Pollak equilibrium. At stage 1, player 2 cannot benefit from deviating to C (since $\frac{8}{7} > 1$), for he myopically places a relative weight $\frac{1}{3}$ on the stage 1 payoff. But notice that the relative weight he attaches at stage 0 to stage 1's continuation payoff equals $\frac{6}{7}$, not $\frac{2}{3}$. As a result, if player 2, given his preferences at stage 0, plans to switch to C at stage 1, then his stage-0 expected payoff rises to $\frac{4}{3}$. Thus, this strategy profile is not a sincere SPE.

Inspired by this example, we now show that, unlike the single-agent case, we can always support a Strotz–Pollak equilibrium path by a strategy profile that meets the stronger requirement of sincere subgame perfection; in other words, there is no conflict between current and future "selves" of any one player.

We shall say that a Strotz–Pollak equilibrium f obeys the *punishment property* if, for any player $i \in N$, history $h \in H$, and action $a_i \in A_i$, we have

$$c_i(f^{|h}|\delta) \geqslant c_i((a_i, f_{-i}(h)), f^{|\{h,(a_i, f_{-i}(h))\}}|\delta).$$

One normally thinks of deviations as securing immediate gain for later punishment. In fact, the opposite might sustain incentive compatibility—trading future gain against current losses. When this is not true, the punishment property obtains.

This property is violated in Example 1. Player 2's continuation payoff in stage 1 is $\frac{5}{7}$ if he plays D, and 1 if he deviates and plays C. It is easy to check that if we replace $\kappa(C, C) = (1, 1)$ by $(0, 0)$ in stage 1, then the resulting Strotz–Pollak equilibrium obeys the punishment property and is sincere subgame perfect. Indeed,

Theorem 2 (*Punishment property*). *If $f \in \Sigma^{S\&P}(\beta, \delta)$ has the punishment property, then it is a sincere SPE.*

The proof is in the appendix, but the intuition is as follows. Everyone assigns more relative weight to current than continuation payoffs—i.e. $(1 - \Delta(\beta, \delta))$ versus $(1 - \delta)$—than he did at any earlier stage. Thus, player i's incentive constraints are met if δ replaces $\Delta(\beta, \delta)$, given the punishment property. This eliminates one-shot deviations planned at a "future stage," as in Example 1. By induction, no one can benefit by deviating in any finite number of stages, or infinitely often, by standard reasoning. Thus, if there is never any temptation to defect this period, as in a Strotz–Pollak equilibrium, then no one ever wishes to *plan to* defect in the future.

Notwithstanding Theorem 2, we now assert that any Strotz–Pollak equilibrium *outcome path* can be supported as a sincere subgame perfect one.

Corollary 1 (*Strotz–Pollak and SPE*). *Every Strotz–Pollak equilibrium path is also a sincere SPE path. Every Strotz–Pollak payoff vector is a sincere SPE vector.*

The argument, in the appendix, is straightforward. By compactness of $U(\beta, \delta)$ and $C(\beta, \delta)$, there exists a worst Strotz–Pollak equilibrium payoff for each player. As in Abreu [1], any Strotz–Pollak equilibrium path can, without loss of generality, be supported by threatening to switch to a player's worst such equilibrium if he deviates. Since such an optimal penal code awards a player the lowest equilibrium continuation payoff if he deviates, it clearly satisfies the punishment property.

Quite unlike the single-agent literature, we can therefore always structure Strotz–Pollak equilibria so that, given the other players' strategies, no conflict arises between the different selves of a player about the optimality of his strategy.

An intuitive explanation of this insight is as follows. In an infinitely repeated game, from the point of view of a player, the strategy profile of the other players defines an infinite-horizon decision problem. Different actions today may lead to different paths, and thereby to different payoffs from tomorrow onward. Thus, given the other players' strategies, a player solves a *standard* single-agent infinite-horizon decision problem but with β–δ preferences, and its associated time inconsistency issues: Namely, a player's optimal strategy *need not* coincide with the one that he would commit to use in the future (were he able). Example 1 depicts *exactly* this. The strategy of player 1 determines a Markov decision problem for player 2 with the following transitions: An action different from D in stage 0 or different from C in stages 2 onwards triggers a switch to a path with value 0, while an action different from D in stage 1 leads to a path with value 1. We saw that the optimal solution of player 2's decision problem exhibits time inconsistency.

But in our environment, the Markov decision problem is not exogenously given. For we can always support a Strotz–Pollak equilibrium path by switching to the worst possible equilibrium payoff for any deviant player. Consequently, the decision problem confronting each player can be assumed to meet an extra "consistency" property—that there is *no conflict* between current and future selves of an agent. For instance, the time inconsistency of player 2 in Example 1 disappears if deviations at any time from $D, D, C, C, \ldots$ trigger a switch to the path with value 0.

It is helpful to emphasize that this is a *result* of our analysis, and *precisely* owes to the repeated game structure we analyze. This cannot be assumed for a general single-agent problem. Our "states" and "transition probabilities" of the infinite-horizon decision problems are *constructed* rather than *given*. Our analysis shows how the simple strategy profile and optimal penal code characterization of Abreu [1] ameliorates the adverse consequences of β–δ preferences in a strategic setting.

The next caveat is instructive. If we posited future-biased preferences, with $\beta > 1$ (for which our model is still well-defined), then Theorem 2 would fail. A reverse of the punishment property would be needed: Players who deviate should expect to be rewarded in the future, and punished immediately. But there is no general way of supporting subgame perfect equilibria failing the punishment property.

3.3. Public randomization and convex payoff sets

We now extend the results, adding a public randomization device to the stage game. This extension critically convexifies the sets of equilibrium and continuation payoffs.

A public randomization device P is a machine that, at the outset of each stage, randomly selects some $p \in [0, 1]$ according to the uniform distribution and publicly informs the players of the realization. In such games, the stage k history $h^k \in H^k$ includes all past actions and public signals. Let $G_P^\infty(\beta, \delta)$ denote the game $G^\infty(\beta, \delta)$ extended by the public randomization device P. In $G_P^\infty(\beta, \delta)$, a *stage-k* strategy for player i is a Borel map f_i from $H^k \times [0, 1]$ to A_i. Player i's expected average discounted payoff, conditional on any history $h \in H = \bigcup_{k=0}^\infty H^k$, then obeys

$$u_i^P(f^{|h}|\beta, \delta) = \int_0^1 [(1 - \Delta(\beta, \delta))\pi_i(f(h, p)) + \Delta(\beta, \delta)c_i^P(f^{|\{h,(f(h,p),p)\}}|\delta)]\, dp,$$

where

$$c_i^P(f^{|h}|\delta) = \int_0^1 [(1 - \delta)\pi_i(f(h, p)) + \delta c_i^P(f^{|\{h,(f(h,p),p)\}}|\delta)]\, dp.$$

Strotz–Pollak equilibrium and sincere SPE easily extend to $G_P^\infty(\beta, \delta)$. Let $U_P(\beta, \delta)$ and $C_P(\beta, \delta)$ be the sets of Strotz–Pollak equilibrium and continuation payoffs.

Fix $\Delta \in (0, 1)$ and $W \subset \mathbb{R}^n$. Let $a(p) : [0, 1] \to A$ be Borel measurable and $\kappa(a, p) : A \times [0, 1] \to W$ Borel measurable w.r.t. p for any $a \in A$. The pair (a, κ) is called (Δ, P)-*admissible* w.r.t. W if, for all $a_i' \in A_i$, $p \in [0, 1]$, and $i \in N$,

$$(1 - \Delta)\pi_i(a_i(p)) + \Delta\kappa(a_i(p), p) \geqslant (1 - \Delta)\pi_i(a_i', a_{-i}(p)) + \Delta\kappa((a_i', a_{-i}(p)), p).$$

For any set $W \subset \mathbb{R}^n$, the set $B_P(W|\beta, \delta)$ is defined by [2]

$$B_P(W|\beta, \delta) = \left\{ v \in \mathbb{R}^n : v = \int_0^1 [(1 - \delta)\pi(a(p)) + \delta\kappa(a(p), p)]\, dp \right.$$
$$\left. \text{and } (a, \kappa) \text{ is } (\Delta(\beta, \delta), P)\text{-admissible w.r.t. } W \right\}.$$

[2] Since the set A is finite, $\kappa : A \times [0, 1] \to W$ is a Caratheodory function; therefore, $\kappa(a(p), p) : [0, 1] \to W$ is Borel measurable for any Borel measurable $a(p) : [0, 1] \to A$ (see [4], for example). Thus, the integral is well-defined.

A set $W \subset V$ is now *self-generating* if $W \subset B_P(W|\beta, \delta)$. We next extend the self-generation and factorization properties of Theorem 1:

Theorem 3 (*Self-generation and factorization*). *Fix* $0 < \beta \leqslant 1$ *and* $\delta \in (0, 1)$.

(a) *If* $W \subset \mathbb{R}^n$ *is a bounded self-generating set, then* $B_P(W|\beta, \delta) \subset C_P(\beta, \delta)$.
(b) Factorization: $C_P(\beta, \delta)$ *is the largest fixed point of the operator* $B_P(\cdot|\beta, \delta)$:

$$C_P(\beta, \delta) = \left\{ v = \int_0^1 [(1-\delta)\pi(a(p)) + \delta\kappa(a(p), p)]\, dp \right.$$
$$\left. \text{and } (a, \kappa) \text{ is } (\Delta(\beta, \delta), P)\text{-admissible w.r.t. } C_P(\beta, \delta) \right\}.$$

Furthermore, $U_P(\beta, \delta) \subset \mathbb{R}^n$ *satisfies*

$$U_P(\beta, \delta) = \left\{ v = \int_0^1 [(1-\Delta(\beta, \delta))\pi(a(p)) + \Delta(\beta, \delta)\kappa(a(p), p)]\, dp \right.$$
$$\left. \text{and } (a, \kappa) \text{ is } (\Delta(\beta, \delta), P)\text{-admissible w.r.t. } C_P(\beta, \delta) \right\}.$$

(c) *The sets* $C_P(\beta, \delta)$ *and* $U_P(\beta, \delta)$ *are convex and compact.*

To see the convexity assertion, note that for any compact set $W \subset \mathbb{R}^n$, any $\beta \in (0, 1]$, and $\delta \in (0, 1)$,

$$B_P(W|\beta, \delta) = co\, B(W|\beta, \delta).$$

Since the convex hull of a compact set is compact, the proof of the compactness of $C_P(\beta, \delta)$ is similar to that of Theorem 4 in APS.

The logic underlying Corollary 1 extends here as well:

Corollary 2 (*Strotz–Pollak and SPE*). *For any* $0 < \beta \leqslant 1$ *and* $0 < \delta < 1$, *every Strotz–Pollak equilibrium payoff vector of* $G_P^\infty(\beta, \delta)$ *can be attained in a sincere SPE.*

The next result is the central tool in the paper for all comparative statics.

Lemma 1 (*Nested generated sets*). *Let* $\beta_1 < \beta_2$ *and* $\delta_1 > \delta_2$ *with* $\Delta(\beta_1, \delta_1) = \Delta(\beta_2, \delta_2)$. *If* $W \subset B_P(W|\beta_1, \delta_1)$ *is compact, then* $B_P(W|\beta_1, \delta_1) \subset B_P(W|\beta_2, \delta_2)$.

Proof. For any $\gamma \in \mathbb{R}^n$, let x_1 maximize the inner product $\gamma \cdot x$ over x in the compact set $B_P(W|\beta_1, \delta_1)$. Put $\Delta = \Delta(\beta_1, \delta_1)$. Then for (a, κ) (Δ, P)-admissible w.r.t. W, define the expected stage and continuation payoffs $\Pi = \int_0^1 \pi(a(p))\, dp$ and $K = \int_0^1 \kappa(a(p), p)\, dp$. Then $x_1 = (1-\delta_1)\Pi + \delta_1 K$. Define $x_2 = (1-\delta_2)\Pi + \delta_2 K$. Since $\Delta(\beta_2, \delta_2) = \Delta$ from the premise of Lemma 1, we also have $x_2 \in B_P(W|\beta_2, \delta_2)$.

Since $K \in B_P(W|\beta_1, \delta_1)$, we have $\gamma \cdot K \leqslant \gamma \cdot x_1$, by the choice of x_1. Given $x_1 = (1-\delta_1)\Pi + \delta_1 K$ and $\gamma \cdot K \leqslant \gamma \cdot x_1$, it follows that $\gamma \cdot \Pi \geqslant \gamma \cdot x_1$. But since $\delta_1 > \delta_2$, we have $\gamma \cdot x_2 = \gamma \cdot [(1-\delta_2)\Pi + \delta_2 K] \geqslant \gamma \cdot [(1-\delta_1)\Pi + \delta_1 K] = \gamma \cdot x_1$. To wit, $B_P(W|\beta_2, \delta_2)$ pushes farther in the γ direction than $B_P(W|\beta_1, \delta_1)$. As γ was arbitrary, any half space containing $B_P(W|\beta_2, \delta_2)$ contains $B_P(W|\beta_1, \delta_1)$. Since a closed convex set is the intersection of the half spaces containing it, $B_P(W|\beta_2, \delta_2) \supset B_P(W|\beta_1, \delta_1)$. □

For a suggestive intuition, note that since $\int_0^1 \kappa(a(p), p)\,dp \in co(W) \subset B_P(W|\delta, \beta)$, the set $B_P(W|\delta, \beta)$ only "expands" further beyond $co(W)$, for fixed $\Delta(\beta, \delta)$, if the weight $1 - \delta$ on current payoffs rises.

The above argument is novel, and critically differs from the proof in APS for standard repeated games. Namely, Theorem 6 in Abreu et al. [3] fixes the action profile played in the current stage game, and then adjusts the continuation payoffs. Any admissible pair supportable at one discount factor is also supportable at a greater discount factor. This logic fails here, and in fact, their approach would yield a puzzling counter-intuition. For when $\delta_1 > \delta_2$ and $\Delta(\beta_1, \delta_1) = \Delta(\beta_2, \delta_2)$, the smaller set $B_P(W|\beta_1, \delta_1)$ places an even greater weight on continuation payoffs than the larger set $B_P(W|\beta_2, \delta_2)$.

When $\beta = 1$, we have $C_P(1, \delta) = U_P(1, \delta)$. A crucial difference, and the source of much novelty here, is that $C_P(\beta, \delta)$ is often a proper subset of $U_P(\beta, \delta)$ if $\beta < 1$. That continuation payoffs are a subset of equilibrium payoffs follows from Lemma 1.

Theorem 4 (*Nested payoff sets*). *For any* $\beta, \delta \in (0, 1)$, $C_P(\beta, \delta) \subset U_P(\beta, \delta)$.

Proof. In Lemma 1, set $\beta_1 = \beta$, $\delta_1 = \delta$, $\delta_2 = \Delta(\beta, \delta) < \delta_1$, $\beta_2 = 1$, and choose $W = C_P(\beta, \delta)$. Observe that with this choice, $\Delta(\beta_1, \delta_1) = \Delta(\beta_2, \delta_2) = \Delta$, say. Since $C_P(\beta, \delta) = B_P(C_P(\beta, \delta)|\beta, \delta)$ and $U_P(\beta, \delta) = B_P(C_P(\beta, \delta)|1, \Delta)$, by Theorem 3(b), Lemma 1 yields the desired inclusion $C_P(\beta, \delta) \subset U_P(\beta, \delta)$. □

Example 2 in Section 4 illustrates that the inclusion in Theorem 4 can be strict.

4. The cost of the present-time bias

4.1. Monotonicity of the equilibrium payoff set?

In the comparative static result of APS, the set of SPE payoffs expands in the discount factor δ. This result accords well with intuition, as it is natural to surmise that players should be able to support more equilibrium outcomes as they grow more patient and thereby value future payoffs more heavily. The recursive techniques of APS afford a simple yet elegant proof of this insight.

We now revisit this issue when $\beta < 1$. Unlike APS, more than one comparative static exercise is possible here. We explore how the set of Strotz–Pollak equilibrium payoffs changes when β or δ change. Theorem 4 complicates the analysis. For changes in either β or δ can differentially affect the sets $C_P(\beta, \delta)$ and $U_P(\beta, \delta)$, and this wrecks havoc on the APS monotonicity intuition.

We first look at the relative importance of β and δ by studying repeated games $G_P^\infty(\beta, \delta)$ with identical incentive constraints, i.e. with $\Delta(\beta, \delta)$ fixed. The next result reveals an interesting trade-off between β and δ. If β rises and δ falls, but the effective discount factor Δ is held fixed, then the equilibrium payoff set expands. Likewise, an "inverse" monotonicity in δ obtains: For fixed $\Delta(\beta, \delta)$, the equilibrium payoff set shrinks in δ. This is yet another application of Lemma 1.

Proposition 1 (*Compensated β–δ changes*). *Let* $\beta_1, \delta_1, \delta_2 \in (0, 1)$ *as well as* $\beta_1 < \beta_2 \leqslant 1$ (*and so* $\delta_1 > \delta_2$). *If* $\Delta(\beta_1, \delta_1) = \Delta(\beta_2, \delta_2)$, *then* $U_P(\beta_1, \delta_1) \subset U_P(\beta_2, \delta_2)$.

Proof. Let $W = C_P(\beta_1, \delta_1)$. Then by Theorem 3(b), $W \subset V$ is compact and obeys $W = B_P(W|\beta_1, \delta_1)$. Using Lemma 1, $W \subset B_P(W|\beta_2, \delta_2)$. So $C_P(\beta_1, \delta_1) \subset B_P(C_P(\beta_1, \delta_1)|\beta_2, \delta_2)$. Then $C_P(\beta_1, \delta_1) \subset C_P(\beta_2, \delta_2)$ by Theorem 3(b).

	L	M	R
U	$1, -1$	$-1, 0$	$-1, 0$
M	$1, -1$	$1, 1$	$1, -1$
D	$-1, 0$	$-1, 0$	$1, \frac{2}{11}$

Fig. 2. Stage game for Example 2.

Finally, define $\Delta = \Delta(\beta_1, \delta_1) = \Delta(\beta_2, \delta_2)$. Anything (Δ, P)-admissible w.r.t. $C_P(\beta_1, \delta_1)$ is (Δ, P)-admissible w.r.t. $C_P(\beta_2, \delta_2)$. So $U_P(\beta_1, \delta_1) \subset U_P(\beta_2, \delta_2)$. □

We now turn to some comparative static analyses in β or δ. Following APS, one might expect that the set of equilibrium payoffs weakly expands in δ. Surprisingly, this well-known result from standard repeated games fails when $\beta < 1$, as we show by example below. But we first give a consequence of Proposition 1, namely, that when monotonicity in δ obtains, the reverse monotonicity holds for β.

Corollary 3 (*Nested payoffs*). *If $U_P(\beta, \delta_2) \subset U_P(\beta, \delta_1)$ for any $\beta \in (0, 1)$ and any $0 < \delta_2 < \delta_1 < 1$, then $U_P(\beta_1, \delta) \subset U_P(\beta_2, \delta)$ for any $\delta \in (0, 1)$ and $0 < \beta_1 < \beta_2 \leqslant 1$.*

Proof. Let $\beta_1 < \beta_2 \leqslant 1$ and choose $\delta_2 < \delta_1$ with $\Delta(\beta_1, \delta_1) = \Delta(\beta_2, \delta_2)$. Then $U_P(\beta_1, \delta_1) \subset U_P(\beta_2, \delta_2)$ by Proposition 1, and $U_P(\beta_2, \delta_2) \subset U_P(\beta_2, \delta_1)$, if $\delta_2 < \delta_1$. □

The next example based on the game in Fig. 2 shows that $U_P(\beta, \delta)$ is not monotonic in β; by the contrapositive of Corollary 3, it is not monotonic in δ either.

Example 2 (*Non-monotonicity in either discount factor*). The minimax point of the game in Fig. 2 is $(1, 0)$. Player 1 can only earn payoff 1 in equilibrium, and so the vertical line segment $[(1, 0), (1, 1)]$ in $\mathbb{R}^2$ contains both $U_P(\beta, \delta)$ and $C_P(\beta, \delta)$. Since (D, R) and (M, M) are Nash equilibria of the stage game, the payoff vectors $(1, \frac{2}{11})$ and $(1, 1)$ belong to $C_P(\beta, \delta) \cap U_P(\beta, \delta)$, for all $\beta, \delta \in (0, 1)$.

Let $\delta = \frac{7}{10}$ and $\beta_1 = \frac{11}{21}$, so that $\Delta(\beta_1, \delta) = \frac{11}{20}$. We first show that player 2's lowest Strotz–Pollak equilibrium continuation payoff vector is $(1, \frac{2}{11})$. Let $Y = [(1, 0), (1, \frac{2}{11}))$. By contradiction, assume $Y \cap C_P(\frac{11}{21}, \frac{7}{10}) \neq \emptyset$. For this to be possible, there must exist a continuation payoff vector $\kappa = (\kappa_1, \kappa_2) \in Y$ supported by an action profile with the stage-game payoff vector $(1, -1)$. But this is impossible on incentive compatibility grounds for player 2. For $\frac{3}{10}(-1) + \frac{7}{10}\kappa_2 < \frac{2}{11}$ implies $\kappa_2 < \frac{53}{77}$, while $\frac{9}{20}(-1) + \frac{11}{20}\kappa_2 \geqslant 0$ (namely, player 2's minimax payoff level) implies $\kappa_2 \geqslant \frac{63}{77}$. So, without loss of generality, we can assume that any deviation by player 2 is punished by the continuation payoff vector $(1, \frac{2}{11})$.

It is not difficult to see that the action profile (U, L) and the continuation value function κ with $\kappa(U, L) = (1, 1)$ and $\kappa(a_1, a_2) = (1, \frac{2}{11})$ for all $(a_1, a_2) \neq (U, L)$ is $\Delta(\beta_1, \delta)$-admissible, with the corresponding equilibrium payoff vector $(1, \frac{1}{10})$. The admissibility of this pair is simple: if player 2 deviates to M or R, his expected payoff equals $\frac{1}{10}$. Therefore, $(1, \frac{1}{10}) \in U_P(\frac{11}{21}, \frac{7}{10})$. Also, it is easy to show that $\frac{1}{10}$ is the lowest equilibrium payoff for player 2, so that $U_P(\frac{11}{21}, \frac{7}{10}) = [(1, \frac{1}{10}), (1, 1)]$.

For an illustration of Theorem 4, observe how

$$C_P\left(\frac{11}{21}, \frac{7}{10}\right) = \left[\left(1, \frac{2}{11}\right), (1, 1)\right] \subsetneq \left[\left(1, \frac{1}{10}\right), (1, 1)\right] = U_P\left(\frac{11}{21}, \frac{7}{10}\right).$$

Let $\beta_2 = \frac{12}{21}$. The following argument shows that $(1, \frac{1}{10}) \notin U_P(\beta_2, \delta)$.

- Observe that $\Delta(\beta_2, \delta) = \frac{4}{7}$ and that $C_P(\frac{11}{21}, \frac{7}{10}) = C_P(\frac{12}{21}, \frac{7}{10})$. Thus, the worst continuation payoff for player 2 is the same as before.
- Any pair (a, κ) that delivers $\frac{1}{10}$ to player 2 uses the action profile (U, L) and thus is not $\Delta(\beta_2, \delta)$-admissible: player 2 earns $\frac{8}{77}$ by deviating from (U, L).
- The payoff vector $(1, \frac{8}{77})$ can be supported by the action profile (U, L) and continuation payoffs $\kappa(U, L) = (1, \frac{41}{44})$, $\kappa(a_1, a_2) = (1, \frac{2}{11})$ for all $(a_1, a_2) \neq (U, L)$.
- It is easy to see that $\frac{8}{77}$ is the lowest equilibrium payoff for player 2. Thus, $U_P(\frac{12}{21}, \frac{7}{10}) = [(1, \frac{8}{77}), (1, 1)]$ and $(1, \frac{1}{10}) \notin U_P(\beta_2, \delta_2)$.

Since $U_P(\frac{12}{21}, \frac{7}{10}) = [(1, \frac{8}{77}), (1, 1)]$ is a proper subset of $U_P(\frac{11}{21}, \frac{7}{10}) = [(1, \frac{1}{10}), (1, 1)]$, the equilibrium payoff set shrinks in β.[3]

The key to understanding the example is found in Theorem 4: Unlike with geometric discounting, the worst possible continuation payoff for a player need not coincide with the worst possible equilibrium payoff. Since (i) the increase in β does not affect the worst possible continuation payoff for player 2 in the example, and (ii) the worst possible equilibrium payoff for him must be supported in both cases by playing (U, L) in the current period (payoff -1), it easily follows that the worst possible equilibrium payoff must increase in this case.

It is natural to explore whether the monotonicity property holds when the worst possible equilibrium and continuation payoffs coincide. That $C_P(\beta, \delta)$ is actually monotone in β and in δ reinforces the plausibility of this conjecture—a result we show in Appendix A.4 (see Claim 2).

An oft-studied class of games in which the worst possible equilibrium and continuation payoffs coincide are those where the minimax point is a pure strategy Nash equilibrium payoff of the stage game. The Prisoner's Dilemma is one such example. With this extra assumption, the equilibrium payoff set is once again monotone.

Proposition 2 (*Uncompensated β–δ changes*). *Assume that the minimax point of the stage game G is a pure strategy Nash outcome of G.*

(a) *If* $\beta, \delta_1, \delta_2 \in (0, 1)$ *and* $\delta_1 < \delta_2$, *then* $U_P(\beta, \delta_1) \subset U_P(\beta, \delta_2)$.
(b) *If* $\beta_1, \beta_2, \delta \in (0, 1)$ *and* $\beta_1 < \beta_2$, *then* $U_P(\beta_1, \delta) \subset U_P(\beta_2, \delta)$.

The proof of part (a) is found in Appendix A.4. Corollary 3 then yields part (b).

4.2. Bounds on the equilibrium payoff set

We now turn to the analysis of the following question: Given β and δ, can we bound $U_P(\beta, \delta)$ by sets of equilibrium payoffs for standard repeated games? This inquiry is important. First, it offers a different perspective on how severe is the failure of the monotonicity property, since geometric discounting is the natural benchmark. How much does the payoff set shrink with $\beta < 1$? Second, it might produce a proof of the folk theorem for β–δ preferences from the standard one with geometric discounting.

Setting $\beta_2 = 1$ in Proposition 1 yields an easy upper bound on the payoff set.

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[3] This example has been chosen for its simplicity. See Appendix A.1 for an example where the same non-monotone behavior occurs with efficient equilibrium payoffs.

	L	R
U	$1, -1$	$-1, 0$
D	$1, 1$	$0, 1$

Fig. 3. Stage game for Example 3.

Proposition 3 (*Upper bound*). *For any $\beta, \delta \in (0, 1)$, $U_P(\beta, \delta) \subset U_P(1, \Delta(\beta, \delta))$.*

Observe that $\beta\delta$ is the largest discount factor that uniformly discounts payoffs by less than the β–δ decision maker:

$$(1, \beta\delta, (\beta\delta)^2, (\beta\delta)^3, \ldots) \leqslant (1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots).$$

Is $U_P(1, \beta\delta)$ then a lower bound on $U_P(\beta, \delta)$? Despite its intuitive appeal, the conjecture is generally wrong: Besides disproving the conjecture, the example also reveals that a slightly smaller β cannot always be offset by greater δ. This shows that the search for a lower bound set may prove a daunting task.

Example 3 (*Payoff lower bound: $U_P(\beta, \delta)$ need not contain $U_P(1, \beta\delta)$*). The minimax point of the stage game in Fig. 3 is $(0, 0)$. Let us show that the set $W = \{(1, 0), (1, 1), (0, 0)\}$ is self-generating if $\beta = 1$ and $\delta = \frac{1}{2}$.

- The payoff vector $(1, 0)$ is supported by the action profile (U, L) and continuation values $(\kappa(U, L), \kappa(U, R), \kappa(D, L), \kappa(D, R)) = ((1, 1), (0, 0), (0, 0), (0, 0))$.
- The payoff vector $(0, 0)$ is supported by the action profile (U, R) and the continuation values $((0, 0), (1, 0), (0, 0), (0, 0))$.
- The payoff vector $(1, 1)$ is a Nash equilibrium outcome of the stage game.

Since W is self-generating, we have $W \subset U(1, \frac{1}{2})$, and, in particular, $(0, 0) \in U(1, \frac{1}{2})$.

We claim that $(0, 0) \notin U(\beta, \delta)$ for any $\beta, \delta \in (0, 1)$.[4] By contradiction, assume that $(0, 0) \in U(\beta, \delta)$ for some $\beta, \delta \in (0, 1)$. Because any continuation payoff vector lies in the (non-negative) set of feasible and individually rational payoffs, only the action profile (U, R) can support the payoff vector $(0, 0)$, with the corresponding continuation payoff vector $\kappa(U, R)$ in the half-open interval $((0, 0), (1, 0)]$. So $\{x \geqslant 0 : (x, 0) \in C(\beta, \delta)\}$ is non-empty, and its maximum x^* exists by compactness of $C(\beta, \delta)$. The continuation payoff vector $\kappa^* = (x^*, 0)$ can only be supported with the action profile (U, L), since the incentive constraints cannot hold for any such pair. Assume, by contradiction, that $(x^*, 0)$ can be supported by some $((U, L), \kappa')$. Then

$$0 = (1 - \delta)(-1) + \delta\kappa_2'(U, L),$$

and the corresponding incentive constraint for player 2 is

$$(1 - \Delta(\beta, \delta))(-1) + \Delta(\beta, \delta)\kappa_2'(U, L) \geqslant \Delta(\beta, \delta)\kappa_2'(U, R).$$

Since $\kappa_2'(U, R) \geqslant 0$, we would have

$$(1 - \Delta(\beta, \delta))(-1) + \Delta(\beta, \delta)\kappa_2'(U, L) \geqslant 0,$$

[4] This claim is also valid if a public randomization device is added to the game, as the example only uses the fact that continuation payoffs must belong to V.

which contradicts $\Delta(\beta, \delta) < \delta$. Thus, $(0, 0) \notin U(\beta, \delta)$ for any $\beta, \delta \in (0, 1)$. In particular, this holds for any β and δ with $\beta\delta = \frac{1}{2}$. Hence, $U_P(1, \frac{1}{2}) \not\subseteq U_P(\beta, \delta)$.

In light of this example, we cannot deduce a folk theorem for the β–δ case from the truth of the standard one. Namely, the fact that $U_P(1, \psi) \uparrow V$ as $\psi \uparrow 1$ cannot be used to prove $U_P(\beta, \delta) \uparrow V$ as $\beta, \delta \to 1$. For by Example 3, since $U_P(1, \delta)$ is weakly increasing in δ, there does not exist a function $\psi(\beta, \delta)$ such that $\lim_{\beta,\delta\to 1} \psi(\beta, \delta) = 1$ and $U_P(1, \psi(\beta, \delta)) \subset U_P(\beta, \delta)$ for all $\beta, \delta \in (0, 1)$.[5] On the other hand, it is easy to see that the (pure strategy minimax) folk theorem does indeed hold with β–δ preferences—simply using the same strategies of Fudenberg and Maskin [6].

5. Conclusion

The standard environment for time inconsistency has been Markovian decision theory exercises, as in an addiction or consumption context. We have shifted this discussion to the repeated games framework, where we have developed a usable framework for exploring β–δ preferences. One may wonder whether the repeated game context can offer any interesting and surprising implications for β–δ preferences, even though the corresponding decision theory problem does not. We feel that the answer is undeniably 'yes': We prove that the most basic comparative static of the repeated games literature—monotonicity in the discount factor—disappears.[6]

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Appendix A

A.1. Non-monotonic efficient equilibrium payoffs

We now give an example showing that *efficient* equilibrium payoffs can be non-monotonic in either discount factor. In the stage game in Fig. 4, (M, R) and (D, M) are Nash equilibria, and $(\frac{2}{11}, 0)$ is the minimax point.

	L	M	R
U	$\frac{24}{11}, -1$	$-2, 0$	$-2, 0$
M	$-2, 0$	$-2, 0$	$\frac{2}{11}, 1$
D	$-2, 0$	$1, \frac{2}{11}$	$-2, 0$

Fig. 4. Stage game yielding non-monotone efficient payoffs.

[5] While the loss of the minimax point alone does not doom the standard folk theorem, we cannot be sure that strictly individually rational points are lost in other games. We hereby thank a referee.

[6] In standard repeated games, monotonicity only fails in contexts without public randomization. See Mailath et al. [11].

Let $\delta = \frac{7}{10}$, $\beta_1 = \frac{11}{21}$, so that $\Delta(\beta_1, \delta) = \frac{11}{20}$. The Pareto frontier of the feasible set is the line of slope -1 obtained by joining the collinear points $(\frac{2}{11}, 1)$, $(1, \frac{2}{11})$, and $(\frac{24}{11}, -1)$. So the payoff vector $(\frac{13}{11} - \frac{1}{10}, \frac{1}{10}) = (1, \frac{2}{11}) + (\frac{2}{11}, -\frac{2}{11}) + (-\frac{1}{10}, \frac{1}{10})$ belongs to the individually rational portion of this line. The action profile (U, L) along with continuation payoffs $\kappa(U, L) = (\frac{2}{11}, 1)$, and $\kappa = (1, \frac{2}{11})$ for any other action profile is a $\Delta(\beta_1, \delta)$-admissible pair, which yields the equilibrium payoff vector $(\frac{13}{11} - \frac{1}{10}, \frac{1}{10})$.

Next, let $\delta = \frac{7}{10}$, $\beta_2 = \frac{12}{21}$. Then $\Delta(\beta_2, \delta) = \frac{4}{7}$. Any pair (a, κ) that gives $\frac{1}{10}$ to player 2 must use the action profile (U, L). But (U, L) cannot be rendered $\Delta(\beta_2, \delta)$-admissible. To see this, we first observe that $\frac{2}{11}$ is the lowest equilibrium continuation payoff for player 2. Indeed, to support a payoff below $\frac{2}{11}$ for player 2 with an admissible pair (a, κ), we must use either the action profile $a = (U, L)$ or some action profile with payoff $(-2, 0)$. In the first case, the argument in Example 2 dismisses $a = (U, L)$. In the second case, incentive compatibility for player 1 is violated, because the average discounted payoff is not individually rational.

Now, if (U, L) is used to support a payoff $\frac{1}{10}$ for player 2, then he can gain by deviating from L, since $\frac{1}{10} < \frac{3}{7} \cdot 0 + \frac{4}{7} \cdot \frac{2}{11} = \frac{8}{77}$. So $\frac{1}{10}$ is not an equilibrium payoff for player 2 when $\beta_2 = \frac{12}{21}$. Hence, $(\frac{13}{11} - \frac{1}{10}, \frac{1}{10}) \in U_P(\beta_1, \delta)$ but $(\frac{13}{11} - \frac{1}{10}, \frac{1}{10}) \notin U_P(\beta_2, \delta)$.

A.2. Punishment property/SPE: Proof of Theorem 2

Assume, by contradiction, that for some history $h^k \in H$, $f^{|h^k}$ is not a sincere Nash equilibrium of $G^\infty(\beta, \delta)$. WLOG, $h^k = h^0$. Then $u_i(g_i, f_{-i}|\beta, \delta)) > u_i(f|\beta, \delta)$ for some player i and strategy $g_i : H \to A_i$. For any $K \in \{0, 1, \ldots\}$, define the function $g_i^K : H \to A_i$ by

$$g_i^K(h) = \begin{cases} g_i(h) & \text{for all } h \in \bigcup_{j=0}^{K} H^j, \\ f_i(h) & \text{for all } h \in \bigcup_{j=K+1}^{\infty} H^j. \end{cases}$$

Given payoff discounting, $u_i(g_i^K, f_{-i}|\beta, \delta) > u_i(f|\beta, \delta)$ for some $K \in \{1, 2, \ldots\}$. Let $h_g = \{a^0, a^1, \ldots, a^K, \ldots\}$ denote the path of action profiles generated by the strategy profile (g_i^K, f_{-i}), and $h_g^k = \{a^0, \ldots, a^{k-1}\}$, $k \in \{1, \ldots\}$. Player i's stage-K average discounted payoff $u_i((g_i^K, f_{-i})^{|h_g^K}|\beta, \delta)$ can be represented as

$$(1 - \Delta(\beta, \delta))\pi_i(g_i^K(h_g^K), f_{-i}(h_g^K)) + \Delta(\beta, \delta)c_i(f^{|h_g^{K+1}}|\delta).$$

Now, because the strategy profile f is a Strotz–Pollak equilibrium of $G(\beta, \delta)$, the induced profile $f(h_g^K)$ is a Nash equilibrium of the reduced normal form game $(N, (A_i)_{i\in N}, ((1-\Delta(\beta, \delta))\pi_i(\cdot) + \Delta(\beta, \delta)c_i(f^{|\{h_g^K, \cdot\}}|\delta))_{i\in N})$. Therefore,

$$\begin{aligned}&(1 - \Delta(\beta, \delta))\pi_i(f(h_g^K)) + \Delta(\beta, \delta)c_i(f^{|\{h_g^K, f(h_g^K)\}}|\delta) \\ &\quad \geqslant (1 - \Delta(\beta, \delta))\pi_i(g_i^K(h_g^K), f_{-i}(h_g^K)) + \Delta(\beta, \delta)c_i(f^{|h_g^{K+1}}|\delta).\end{aligned}$$

For a contradiction, assume that $g_i^K(h_g^K) \neq f_i(h_g^K)$. Because $\Delta(\beta, \delta) < \delta$, and since $c_i(f^{|\{h_g^K, f(h_g^K)\}}|\delta) \geqslant c_i(f^{|h_g^{K+1}}|\delta)$ by the punishment property, we conclude that

$$(1 - \delta)\pi_i(f(h_g^K)) + \delta c_i(f^{|\{h_g^K, f(h_g^K)\}}|\delta) \geqslant (1 - \delta)\pi_i(g_i^K(h_g^K), f_{-i}(h_g^K)) + \delta c_i(f^{|h_g^{K+1}}|\delta).$$

Therefore, player i improves his stage-0 expected payoff by employing the strategy $f_i(h_g^K)$ instead of $g_i^K(h_g^K)$ at stage K. That is,

$$u_i(g_i^{K-1}, f_{-i}|\beta, \delta) \geqslant u_i(g_i^K, f_{-i}|\beta, \delta).$$

We similarly get the inequality chain $u_i(f|\beta, \delta) \geqslant u_i(g_i^0, f_{-i}|\beta, \delta) \geqslant \cdots \geqslant u_i(g_i^K, f_{-i}|\beta, \delta)$. This contradicts our premise $u_i(g_i^K, f_{-i}|\beta, \delta) > u_i(f|\beta, \delta)$.

A.3. Strotz–Pollak and sincere SPE: Proof of Corollary 1

Let $Q(f)$ denote the infinite sequence of action profiles (the *path*) that results from conformity with the strategy profile $f = (f_1, \ldots, f_n)$ in the absence of deviations, and let $\Omega = A^\infty$ denote the set of paths. For each $Q \in \Omega$, let $Q = \{a^k(Q)\}_{k=0}^\infty$, where $\{a^k(Q)\}_{k=0}^\infty$ is the corresponding sequence of action profiles.

Let $Q_i \in \Omega$, $i = 0, 1, \ldots, n$. Following Abreu [1], let $\sigma(Q_0, Q_1, \ldots, Q_n)$ denote the corresponding *simple strategy profile*. Here Q_0 is the initial path and Q_i is the punishment path for any deviation by player i after any history. Slightly abusing notation, define continuation payoffs from path Q starting at stage k to be

$$\kappa_i(Q, k) = (1-\delta)\sum_{s=0}^{\infty} \delta^s \pi_i(a^{k+s}(Q)).$$

By the definition of a Strotz–Pollak equilibrium, we have:

Claim 1. *A simple strategy profile $\sigma(Q_0, \ldots, Q_n)$ is a Strotz–Pollak equilibrium iff*

$$\begin{aligned}&(1-\Delta(\beta,\delta))\pi_i(a^k(Q_j)) + \Delta(\beta,\delta)\kappa_i(Q_j, k+1)\\ &\quad \geqslant (1-\Delta(\beta,\delta))\pi_i(a_i', a_{-i}^k(Q_j)) + \Delta(\beta,\delta)\kappa_i(Q_i, k+1)\end{aligned}$$

for all actions $a_i' \in A_i$, players $i \in N$, indices $j \in \{0\} \cup N$, and stages $k = 0, 1, \ldots$.

An *optimal penal code* for $G^\infty(\beta, \delta)$ is an n-vector of pure strategy profiles $(\underline{f_1}, \underline{f_2}, \ldots, \underline{f_n})$, where $\underline{f_i} \in \Sigma^{S\&P}(\beta, \delta)$ delivers the worst possible punishment for player i, namely, whose associated continuation payoff is $c_i(\underline{f_i}|\delta) = \min_{v \in \overline{C}(\beta,\delta)} v_i$. Existence of an optimal penal code in $G^\infty(\beta, \delta)$ owes to compactness of $\overline{C}(\beta, \delta)$.

Corollary 1 can now be reformulated: *If $f \in \Sigma^{S\&P}(\beta, \delta)$ and $(\underline{f_1}, \ldots, \underline{f_n})$ is an optimal penal code, then $\sigma = \sigma(Q(f), Q(\underline{f_1}), \ldots, Q(\underline{f_n}))$ is a sincere SPE.* To see this, observe that σ is a Strotz–Pollak equilibrium by Claim 1. Since it has the punishment property, it is a sincere SPE by Theorem 2.

A.4. Comparative statics: Proof of Proposition 2

We first prove the monotonicity of $C_P(\beta, \delta)$ in a proof that parallels that of APS.

Claim 2. (a) *If $\beta, \delta_1, \delta_2 \in (0, 1)$ and $\delta_1 < \delta_2$, then $C_P(\beta, \delta_1) \subset C_P(\beta, \delta_2)$.*
(b) *If $\beta_1, \delta \in (0, 1)$, $\beta_2 \in (0, 1]$, and $\beta_1 < \beta_2$, then $C_P(\beta_1, \delta) \subset C_P(\beta_2, \delta)$.*

Proof of Claim. Let $W \subset \mathbb{R}^n$ be compact with $W \subset B_P(W|\beta, \delta_1)$. We first show that $W \subset B_P(B_P(W|\beta, \delta_1)|\beta, \delta_2)$. Pick any $x \in W$. Since $x \in B_P(W|\beta, \delta_1)$, there exists a pair (a, κ^0) that is $(\Delta(\beta, \delta_1), P)$-admissible w.r.t. *W*, such that

$$x = \int_0^1 [(1 - \delta_1)\pi(a(p)) + \delta_1 \kappa^0(a(p), p)]\, dp.$$

For all $p \in [0, 1]$, we have

$$w(p) \equiv (1 - \delta_1)\pi(a(p)) + \delta_1 \kappa^0(a(p), p) \in B(W|\beta, \delta_1). \tag{2}$$

Let us show that $w(p) \in B(B(W|\beta, \delta_1)|\beta, \delta_2)$ for all $p \in [0, 1]$. Fix $\lambda = \frac{\delta_1(1-\delta_2)}{\delta_2(1-\delta_1)}$. Define the continuation value function $\kappa^1 : A \times [0, 1] \to B(W|\beta, \delta_1)$

$$\kappa^1(a, p) \equiv (1 - \lambda)w(p) + \lambda \kappa^0(a, p). \tag{3}$$

Substituting the $\kappa^0(a, p)$ expression from (3) into Eq. (2) then yields $w(p) = (1 - \delta_2)\pi(a(p)) + \delta_2 \kappa^1(a(p), p)$. For incentive compatibility, we need to verify that

$$(1 - \delta_2)\pi(a(p)) + \beta\delta_2\kappa^1(a(p), p) \geqslant (1 - \delta_2)\pi(a_i, a_{-i}(p)) + \beta\delta_2\kappa^1((a_i, a_{-i}(p)), p)$$

for all $a_i \in A_i$ and $i \in N$. This incentive constraint holds since (a, κ^0) is $(\Delta(\beta, \delta_1), P)$-admissible w.r.t. *W*, and both sides above can be rewritten using the identity

$$(1 - \delta_2)\pi(a) + \beta\delta_2\kappa^1(a, p) = \frac{1 - \delta_2}{1 - \delta_1}[(1 - \delta_1)\pi(a) + \beta\delta_1\kappa^0(a, p)] + \frac{\beta(\delta_2 - \delta_1)}{1 - \delta_1} w(p).$$

Since $B(B(W|\beta, \delta_1)|\beta, \delta_2) \subset B_P(B_P(W|\beta, \delta_1)|\beta, \delta_2)$, we have $w(p) \in B_P(B_P(W|\beta, \delta_1)|\beta, \delta_2)$, and thus $x = \int_0^1 w(p)\, dp \in B_P(B_P(W|\beta, \delta_1)|\beta, \delta_2)$.

In the particular case where $W = C_P(\beta, \delta_1)$, any point of $C_P(\beta, \delta_1)$ belongs to $B_P(C_P(\beta, \delta_1)|\beta, \delta_2)$ since $B_P(C_P(\beta, \delta_1)|\beta, \delta_1) = C_P(\beta, \delta_1)$. Since $C_P(\beta, \delta_2)$ is the largest fixed point of $B_P(W|\beta, \delta_2)$, we conclude that $C_P(\beta, \delta_1) \subset C_P(\beta, \delta_2)$.

The second part of the statement follows from Corollary 3. □

For any $x \in U_P(\beta, \delta_1)$, we will show $x \in U_P(\beta, \delta_2)$. Now, since $x \in U_P(\beta, \delta_1)$, there exists a pair (a, κ^0) that is $(\Delta(\beta, \delta_1), P)$-admissible w.r.t. $C_P(\beta_1, \delta)$ such that

$$x = \int_0^1 [(1 - \Delta(\beta, \delta_1))\pi(a(p)) + \Delta(\beta, \delta_1)\kappa^0(a(p), p)]\, dp.$$

WLOG, we use the least continuation $\kappa_i^0((a_i, a_{-i}(p)), p) = \min_{w \in C_P(\beta,\delta_1)} e_i \cdot w \equiv v_i^*$ for all actions $a_i \neq a_i(p)$, and all $p \in [0, 1]$, $i \in N$, where e_i is a vector with 1 in its *i*th component and 0 otherwise. Recalling Theorem 4, the payoff

$$w(p) = (1 - \Delta(\beta, \delta_1))\pi(a(p)) + \Delta(\beta, \delta_1)\kappa^0(a(p), p),$$

which belongs to $U_P(\beta, \delta_1)$, need not belong to $C_P(\beta, \delta_1)$.

Since $x = \int_0^1 w(p)\, dp$, it suffices that $w(p) \in U_P(\beta, \delta_2)$ for all $p \in [0, 1]$. Define

$$\lambda \equiv \frac{\Delta(\beta, \delta_1)(1 - \Delta(\beta, \delta_2))}{\Delta(\beta, \delta_2)(1 - \Delta(\beta, \delta_1))} = \frac{\delta_1(1 - \delta_2)}{\delta_2(1 - \delta_1)} < 1.$$

For any $p \in [0, 1]$, define the continuation

$$\kappa^1(a, p) = \begin{cases} v^* & \text{if } \exists i \in N : a = (a_i, a_{-i}(p)), a_i \neq a_i(p) \\ (1-\lambda)w(p) + \lambda\kappa^0(a, p) & \text{otherwise.} \end{cases}$$

With this selection, notice that we have

$$w(p) = (1 - \Delta(\beta, \delta_2))\pi(a(p)) + \Delta(\beta, \delta_2)\kappa^1(a(p), p). \tag{4}$$

We must verify that the incentive constraints are met, and $\kappa^1(a, p) \in C_P(\beta, \delta_2)$.

For incentive compatibility, we verify that for all actions $a_i \in A_i$, and all $i \in N$:

$$w_i(p) \geqslant (1 - \Delta(\beta, \delta_2))\pi_i(a_i, a_{-i}(p)) + \Delta(\beta, \delta_2)v_i^*. \tag{5}$$

By contradiction, assume that (5) fails for some player $i \in N$, and let his most profitable deviation be action $a_i' \neq a_i(p)$. Then

$$w_i(p) < (1 - \Delta(\beta, \delta_2))\pi_i(a_i', a_{-i}(p)) + \Delta(\beta, \delta_2)v_i^*. \tag{6}$$

Comparing (4) and (6), given $\kappa^1(a(p), p) \geqslant v_i^*$, we get $\pi_i(a_i', a_{-i}(p)) > \pi_i(a(p))$.

Next, $w_i(p) \geqslant (1 - \Delta(\beta, \delta_1))\pi_i(a_i, a_{-i}(p)) + \Delta(\beta, \delta_1)v_i^*$ for all $a_i \in A_i$, since $w(p) \in U_P(\beta, \delta_1)$. Comparing this inequality with (6), and using $\Delta(\beta, \delta_1) < \Delta(\beta, \delta_2)$, we get $v_i^* > \pi_i(a_i', a_{-i}(p))$. Hence, $v_i^* > \pi_i(a_i, a_{-i}(p))$ for all $a_i \in A_i$, as a_i' was the most profitable deviation. This contradicts the definition of a minimax payoff, proving (5).

To finish the proof, we must show that $\kappa^1(a(p), p) \in C_P(\beta, \delta_2)$.

Define the line segment $L(\delta)$ from $(1 - \delta)\pi(a(p)) + \delta\kappa^0(a(p), p)$ to $\kappa^0(a(p), p))$. Then $L(\delta_1) \subset C_P(\beta, \delta_1)$ since (a, κ^0) is $(\Delta(\beta, \delta_1), P)$-admissible w.r.t. $C_P(\beta_1, \delta)$, and $\kappa^0 \in C_P(\beta_1, \delta)$, which is a convex set. On the other hand, $L(\delta_2) \subset L(\delta_1)$ because $\kappa^0(a(p), p)) \in C_P(\beta, \delta_1)$ and $\delta_2 > \delta_1$. Finally, we have $L(\delta_2) \subset C_P(\beta, \delta_2)$ by Claim 2.

If $\kappa^1(a(p), p) \in L(\delta_2)$, then we are done. Assume not. For $n = 1, 2, \ldots,$ define

$$\kappa^n(a(p), p) \equiv \pi(a(p)) + [\kappa^1(a(p), p) - \pi(a(p))]/\delta_2^n. \tag{7}$$

Notice that $\kappa^n(a(p), p)$ is on the (extended) line formed from $L(\delta_2)$. For large enough n, $\|\kappa^n(a(p), p) - \pi(a(p))\| \geqslant \delta_2\|\kappa^0(a(p), p) - \pi(a(p))\|$, since the left side explodes in n. Choose the least such n. Then $\|\kappa^n(a(p), p) - \pi(a(p))\| < \|\kappa^0(a(p), p) - \pi(a(p))\|$, for if not, $\|\kappa^{n-1}(a(p), p) - \pi(a(p))\| \geqslant \delta_2\|\kappa^0(a(p), p) - \pi(a(p))\|$. But

$$(1 - \delta_2)\pi(a(p)) + \delta_2\kappa^0(a(p), p) - \pi(a(p)) = \delta_2[\kappa^0(a(p), p) - \pi(a(p))],$$

the line segment $L(\delta_2)$ consists precisely those points on the extended line meeting these two inequalities. In sum, we have shown that $\kappa^n(a(p), p) \in L(\delta_2) \subset C_P(\beta, \delta_2)$.

Let us show that for $m = 1, 2, \ldots, n$, all $a_i \in A_i$, and all $i \in N$, we have

$$(1 - \Delta(\beta, \delta_2))\pi(a(p)) + \Delta(\beta, \delta_2)\kappa^m(a(p), p) \geqslant (1 - \Delta(\beta, \delta_2))\pi(a_i, a_{-i}(p)) + \Delta(\beta, \delta_2)v^*. \tag{8}$$

For $m = 1$, (8) reduces to (5). Assume it true for $m \geqslant 1$. Since $\kappa^{m-1}(a(p), p) = (1-\delta_2)\pi(a(p)) + \delta_2\kappa^m(a(p), p)$ by our definition (7), inequality (8) at $m + 1$ becomes

$$\left(1 - \frac{\Delta(\beta, \delta_2)}{\delta_2}\right)\pi(a(p)) + \frac{\Delta(\beta, \delta_2)}{\delta_2}\kappa^m(a(p), p) \geqslant (1 - \Delta(\beta, \delta_2))\pi(a_i, a_{-i}(p)) + \Delta(\beta, \delta_2)v^*.$$

If $\pi(a(p)) \geqslant \pi(a_i, a_{-i}(p))$, we are done, since $\kappa^m(a(p), p) \geqslant v^*$ and $\Delta(\beta, \delta_2)/\delta_2 < 1$. But if $\pi(a(p)) < \pi(a_i, a_{-i}(p))$, then the result owes to $1 > \Delta(\beta, \delta_2)/\delta_2$ and (8).

Now, $\kappa^n(a(p), p) \in C_P(\beta, \delta_2)$, and (8) for $m = n-1$ implies that $\kappa^{n-1}(a(p), p) \in C_P(\beta, \delta_2)$. Proceeding recursively, $\kappa^m(a(p), p) \in C_P(\beta, \delta_2)$ for all $m = 1, 2, \ldots, n$. Then in particular, $\kappa^1(a(p), p) \in C_P(\beta, \delta_2)$, as required.

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