University of Michigan Economic Theory Prelim
September 2000

Answer all questions; part A in one book, and part B in another. While questions have equal weight, passing requires demonstrating *command* of both parts. Cite any and all relevant theorems. Do not substitute hand-waving for rigorous proofs.

A. Game Theory

1. Consider the following infinitely repeated Cournot-Duopoly game. At the beginning of each period, the firms observe the "state of the market". The state can be either "high" or "low" with equal probability. The inverse demand function is

$$p = 25 - Q$$

when the state is high and

$$p = 13 - Q$$

when the state is low, where Q denotes the total output of the firms. Each period, after observing the state of the market, the firms simultaneously choose production quantities. The firms have the same production technology with constant marginal cost $c = 1$, and the same discount factor δ $(0, 1)$. This is the standard Cournot-Duopoly game, except that the inverse demand function is random. To simplify the analysis, we will only consider the Cournot-Nash reversion punishment. Notice, however, that in this game, this implies that each firm produces output q_H^N when the state is high and q_L^N when the state is low, where q_H^N and q_L^N represent the corresponding one-period Nash equilibrium outputs.

   (a) What is the smallest discount factor δ for which monopoly output in every period can be sustained with the Cournot-Nash reversion punishment?

   (b) Suppose that $\delta = 1/2$. What is the most collusive symmetric path that can be supported with Cournot-Nash reversion?

2. Prove that every correlated equilibrium is rationalizable. By example, show that the converse is false. That is, show that there are rationalizable strategy profiles that get no positive weight in any correlated equilibrium.

NB: A *correlated equilibrium* for a game $G = (S_1, \ldots, S_n; u_1, \ldots, u_n)$ is a probability distribution p over $S = S_1 \times \cdots \times S_n$ such that for each i and s_i, t_i S_i with $\sum_{s_{-i}} p(s_i, s_{-i}) > 0$,

$$\sum_{s_{-i}} p(s_{-i} \mid s_i) u_i(s) \geq \sum_{s_{-i}} p(s_{-i} \mid s_i) u_i(t_i, s_{-i}),$$

where $p(s_{-i} \mid s_i)$ denotes i's conditional probability of s_{-i} given his "signal" s_i. A correlated equilibrium is implemented by a random device that selects each s S with probability $p(s)$ and *privately* informs each player i of his/her coordinate s_i of the random outcome s.

B. Uncertainty Theory

1. Given vectors $x, y \quad \mathbf{R}^n_+$, write $x \quad y$ if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ for any $k = 1, \dots, n$. Given matrices $A, B \quad \mathsf{R}^{n \times m}_+$, write $A \quad B$ if for each row $i = 1, \dots, n$, we have $A_{i,\bullet} \quad B_{i,\bullet}$. Also, the vector x is *nonincreasing* if $x_i \geq x_{i+1}$ for all $i = 1, \dots, n-1$.

   (a) [Warm-up] Prove the following easy extrapolation on the first ranking theorem: Let $f \quad g$, and y is decreasing, where $f, g, y \quad \mathsf{R}^n_+$. Then $\sum_{i=1}^n f_i y_i \leq \sum_{i=1}^n g_i y_i$. Under what additional assumption does this reduce to the first ranking theorem?

   (b) Let $T, \hat{T} \quad \mathsf{R}^{n \times n}_+$ and $Q, \hat{Q} \quad \mathsf{R}^{n \times m}_+$. Assume that $T \quad \hat{T}$, $Q \quad \hat{Q}$, and finally $Q_{i+1,\bullet} \quad Q_{i,\bullet}$ for all $i = 1, \dots, n-1$.

   Use part (b) to prove that $TQ \quad \hat{T}\hat{Q}$.

   Hint: Operate the matrices on vectors $w_k' = (1, \dots, 1, 0, \dots, 0)$, i.e. with k ones.

   (c) Let square matrices $R, \hat{R} \quad \mathsf{R}^{n \times n}_+$ have eigenvalues less than 1. It is well-known that $(I - R)^{-1}$ and $(I - \hat{R})^{-1}$ are both well-defined, eg. $(I - R)^{-1} = I + R + R^2 + \cdots$. Assume that $R \quad \hat{R}$, and $R_{i+1,\bullet} \quad R_{i,\bullet}$ and $\hat{R}_{i+1,\bullet} \quad \hat{R}_{i,\bullet}$ for all $i = 1, \dots, n-1$. Use parts (a) and (b) to show that $(I - R)^{-1} \quad (I - \hat{R})^{-1}$.

   Hint: Use recursion, i.e. $y(0) = w_k$ and for each $r \geq 0$, $y(r+1) = Ry(r) + w_k$. Thus, $\lim_{r \to \infty} y(r) = (I - R)^{-1} w_k$.

2. Testimony has ended, and jurors are cloistered in a conference room, deciding whether to convict or acquit a defendant. As a result of the trial testimony and arguments, each of the n jurors $j = 1, 2, \dots, n$ has received a private signal s_j with range $\{\gamma, \iota\}$, where signal outcome γ favours the state $G = \{$accused is guilty$\}$, and ι favours the state $N = \{$accused is not guilty (i.e. is innocent)$\}$. A priori, the states are equilikely. It is common knowledge that signals are conditionally iid. In state G, $s_j = \gamma$ with chance 2/3, and $s_j = \iota$ with chance 1/3. In state N, $s_j = \iota$ with chance θ, and $s_j = \gamma$ with chance $1 - \theta$. Assume $\theta > 1/2$.

   Assume the jurors' decision mechanism employs a one-shot game: In the conference room, each privately votes 'convict' or 'acquit' using their private signal alone. Majority rule prevails (and n is odd).

   (a) Deduce whether the juror's voting strategies are *sincere* — i.e. each votes to convict iff he receives the signal outcome $s = \gamma$. If your answer depends on θ, give your answer parametrically. Explain intuitively what is going on.

   (b) Assume that there is no jury, but that an unbiased judge (i.e. with a fair prior) alone decides. Assume that the payoff to making the right decision is one, while the payoff of the wrong decision is zero. What is the above signal s worth to him?