

Econ 618: Assignment 1

Note 1: Brevity is appreciated, but be sure not to wave your hands in a proof.

Note 2: New questions are from last year's final exam and the past theory prelim.

1. This question tests if you understand chain, lattice, and poset. Fix a partial order \succ . [5 pts]
Produce a lattice using \succ , such that if you remove a point from it, it is a poset but not a lattice, and if you remove another point from it, it becomes a nontrivial (i.e. more than one element) chain.
Hint: The question is not deep.
2. Let X and Y be lattices and Z a poset. Suppose that $f : X \times Y \times Z \rightarrow \mathfrak{R}$ is a bounded supermodular function in $(x, y) \in X \times Y$. Assume f has increasing differences between any $a = (x, y) \in X \times Y$ and $z \in Z$. Define $g(x, z) \equiv \sup_{y \in Y} f(x, y, z)$. (2 hrs.)
[If sup confuses you or slows you down, try using max (and switch back afterwards).]
 - (a) Warm-up: Show that g supermodular on $x \in X$. [2 pts]
 - (b) Show that g has increasing differences in $(x, z) \in X \times Z$. [13 pts]
3. The SSD stochastic ranking theorem of cdf's F and G we had showed that SSD allows us to order the expectations of any concave and increasing function under F versus G . Using the method of cones from class (you may be as loose as we were in class), find a condition different than SSD that allows you to order the expectations of any convex and increasing functions. (45 min.) [10 pts]
4. Given vectors $x, y \in \mathbf{R}_+^n$, write $x \succeq y$ if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ for any $k = 1, \dots, n$. Given matrices $A, B \in \mathfrak{R}_+^{n \times m}$, write $A \succeq B$ if $A_{i,\bullet} \succeq B_{i,\bullet}$ for each row $i = 1, \dots, n$.
 - (a) [Warm-up] Prove the following easy extrapolation on the first ranking theorem: Let $f \succeq g$, and y is decreasing, where $f, g, y \in \mathfrak{R}_+^n$. Then $\sum_{i=1}^n f_i y_i \leq \sum_{i=1}^n g_i y_i$. Under what additional assumption does this reduce to the first ranking theorem?
 - (b) Let $T, \hat{T} \in \mathfrak{R}_+^{n \times n}$ and $Q, \hat{Q} \in \mathfrak{R}_+^{n \times m}$. Assume that $T \succeq \hat{T}$, $Q \succeq \hat{Q}$, and finally $Q_{i+1,\bullet} \succeq Q_{i,\bullet}$ for all $i = 1, \dots, n-1$.
Use part (b) to prove that $TQ \succeq \hat{T}\hat{Q}$.
Hint: Operate the matrices on vectors $w'_k = (1, \dots, 1, 0, \dots, 0)$, i.e. with k ones.
 - (c) Let square matrices $R, \hat{R} \in \mathfrak{R}_+^{n \times n}$ have eigenvalues less than 1. It is well-known that $(I-R)^{-1}$ and $(I-\hat{R})^{-1}$ are both well-defined, eg. $(I-R)^{-1} = I+R+R^2+\dots$. Assume that $R \succeq \hat{R}$, and $R_{i+1,\bullet} \succeq R_{i,\bullet}$ and $\hat{R}_{i+1,\bullet} \succeq \hat{R}_{i,\bullet}$ for all $i = 1, \dots, n-1$.
Use parts (a) and (b) to show that $(I-R)^{-1} \succeq (I-\hat{R})^{-1}$.
Hint: Use recursion, i.e. $y(0) = x_k$ and for each $r \geq 0$, $y(r+1) = Ry(r) + x_k$. Thus, $\lim_{r \rightarrow \infty} y(r) = (I-R)^{-1}x_k$.

Practice Questions from Past Years' Assignments (not due!)

1. Show that maximization need not preserve submodularity. Give a brief proof that minimization does preserve submodularity.
2. In class we showed that f supermodular on a lattice product implies f has increasing differences, and that if f has increasing differences and is supermodular component-wise (or each component is real) then f is supermodular.
Prove if you can (and if it is true; if not, try to provide a counterexample) the ordinal analogue of this theorem, where quasisupermodularity replaces supermodularity, and the single crossing property replaces increasing differences.
3. We proved that an increasing logsupermodular function is supermodular. Show that the increasing proviso is needed.