Solving Stochastic Differential Equations by the Reduction Method

1. Solve $dX = (\sqrt{1+X^2} + X/2)dt + \sqrt{1+X^2}dW$ **subject to** $X_0 = 0$**.**

Solution: Transform $Y = g(X)$ with a monotonic function g to get

$$\begin{aligned} \Longrightarrow dY &= g_x dX + (1/2)g_{xx}(dX)^2 \\ &= (g_x[\sqrt{1+X^2} + X/2] + (1/2)g_{xx}(1+X^2)])dt + g_x\sqrt{1+X^2}dW \end{aligned}$$

So look for a function g with a constant factor of the Weiner increment:

$$g_x\sqrt{1+x^2} = 1 \Longrightarrow g(x) = \log(x + \sqrt{1+x^2})$$

But then we nicely find that the factor of dt is also constant:

$$g_x[\sqrt{1+X^2} + X/2] + (1/2)g_{xx}(1+X^2)] = 1$$

We then have $dY_t = dt + dW \Longrightarrow Y_t = t + W_t$, since $Y_0 = g(X_0) = g(0) = 0$. Finally, inverting we have

$$X_t = g^{-1}(Y_t) = sinh(Y_t) = sinh(t + W_t),$$

where $sinh(z) = (e^z - e^{-z})/2$ is the inverse of g, also known as the hyperbolic sine.

2. Solve: $dX = f(X)dt + cXdW$ **and** $X_0 = 1$ **for** $f(x) = ax\log|x| + bx$**.**

Solution: Let's try transforming $Y = g(X)$, with g increasing.

$$\Longrightarrow dY = g_x dX + (1/2)g_{xx}(dX)^2 = [fg_x + (1/2)c^2X^2g_{xx}]dt + g_x cXdW$$

Try $g(x) = \log|x|$. In fact, the solution will only have $X_t > 0$, so that $|\ldots|$ is not needed. Then

$$\begin{aligned} dY &= [(aX\log|X| + bX)1/X + (1/2)c^2X^2(-1/X^2)]dt + [1/X]cXdW \\ &= [a\log|X| + b - c^2/2]dt + cdW \end{aligned}$$

Well, we haven't hit a reducible function yet. But just as one does for standard differential equations, let's use the method of variation of parameters. We will remove the homogeneous (non-constant) portion of the SDE, namely, $d\hat{Y}' = a\hat{Y}dt$, whose solution is $\hat{Y}_t = e^{at}$. Define $Y_t \equiv e^{at}Z_t \Longrightarrow Z_t = e^{-at}Y_t$. Then we have

$$dZ_t = d(e^{-at}Y_t) = -ae^{-at}Ydt + e^{-at}dY = (b - c^2/2)e^{-at}dt + ce^{-at}dW$$

$$\begin{aligned} \Longrightarrow Z_t &= (b - c^2/2)[1 - e^{-at}]/a + c\int_0^t e^{-as}dW_s \\ &= (b - c^2/2)[1 - e^{-at}]/a + c[e^{-at}W_t + a\int_0^t e^{-as}W_s ds] \\ \Longrightarrow Y_t &= e^{at}Z_t = (b - c^2/2)[e^{at} - 1]/a + cW_t + ac\int_0^t e^{a(t-s)}W_s ds \\ \Longrightarrow X_t &= e^{Y_t} = \exp\left((b - c^2/2)[e^{at} - 1]/a + cW_t + ac\int_0^t e^{a(t-s)}W_s ds\right) \end{aligned}$$

Finally, you can check the solution in various ways. You can differentiate it entirely, and remember to use Ito's Rule (!), and see that it solves the original problem. A quicker check that won't reveal all problems is to set $c = 0$ [kill the noise!] and get that

$$X_t = \exp\left(b[e^{at} - 1]/a\right)$$

By differentiation, this solves

$$dX_t = bX_te^{at}dt = bX_t + aX_t[(b/a)(e^{at} - 1)]dt = [bX_t + aX_t\log X_t]dt = f(X_t)dt$$

as needed. You can also try $a = 0$ (by a l'Hôpital limit) to get that $X_t = \exp{(b - c^2/2)t + cW_t}$. This makes sense, hopefully, in light of our formula for linear SDE's.