Computing the Diffusion Process Limit of a Sequence of Suitably Scaled Continuous Time Stochastic Processes

by Lones Smith (2001)

Question: Let $\{X^N(t)\}$ be a stochastic process with deterministic continuous changes satisfying $dX^N/dt = \alpha X^N$. There are also periodic random jumps in any interval $(t, t+1/N)$ to

$$\begin{cases} \beta X^N, & \text{with chance } \lambda/2N + o(1/N) \\ (2-\beta)X^N, & \text{with chance } \lambda/2N + o(1/N) \end{cases}$$

Now, let β and γ be parameterized by N, with $\beta_N \to 1$ and $\lambda_N \to \infty$ such that $\lambda_N(1-\beta_N)^2 = \gamma$ for some constant $\gamma > 0$.

Explain why $X^N \to^{\mathcal{D}} Y$, where Y_t is a diffusion obeying the stochastic differential equation $dY_t = \alpha Y_t dt + \gamma Y_t^2 dW_t$. In other words, the drift owes to the deterministic part, and the volatility to the stochastic jumps.

Comments: In general, you should be able to *deduce* the limit, without knowing it. Observe that this knowledge is inessential below, because we compute it.

Crucially, the Poisson jumps shrink in size to zero as N explodes. Otherwise, $\{X^N(t)\}$ could not possibly converge in distribution upon a process with continuous sample paths, which is true of an SDE like $\langle Y_t \rangle$.

Solution: Since $\{X^N(t)\}$ is a family of continuous time stochastic processes, let's just compute the needed moments, and claim the general convergence theorem asserted in class, taken from Karlin-Taylor, volume II, chapter 15.

1. Drift: Adding together the continuous and random changes in X^N:

$$\begin{aligned} E[(\Delta X^N)|X^N(t)=x] &= (e^{\alpha/N}-1)x + [\lambda/2N + o(1/N)](\beta_N - 1)x \\ &\quad + [\lambda/2N + o(1/N)](1-\beta_N)x \\ &= \alpha x/N + o(1/N) \end{aligned}$$

where we have used the known fact that $e^y = 1 + y + y^2/2 + \cdots = 1 + y + o(y)$. Finally, this is of the form $\alpha\delta x + o(\delta)$, for all $\delta = 1/N > 0$ length intervals, as required.

2. Variance: Analogously, by collecting the second two terms, we have

$$\begin{aligned} E[(\Delta X^N)^2|X^N(t)=x] &= (e^{\alpha/N}-1)^2x^2 + [\lambda_N/N + o(1/N)](\beta_N-1)^2x^2 \\ &= (\alpha/N)^2x^2 + o(1/N^2) + \lambda_N(\beta_N-1)^2x^2/N + (\beta_N-1)^2 o(1/N) \\ &= \gamma x^2/N + o(1/N) \end{aligned}$$

which is of the form $\gamma\delta x^2 + o(\delta)$, for all $\delta > 0$ length intervals, as required.

3. Higher Moment of ΔX^N: In particular, let's try the third moment. As in the variance computation, the dominant term contribution to $E[(\Delta X^N)^3|X^N(t)=x]$ will be $[\lambda_N/2N + o(1/N)](1-\beta_N)^3 = \lambda_N(1-\beta_N)^3/N = (1-\beta_N)\gamma/N$, and the factor on $1/N$ will vanish because $\beta_N \to 1$. In other words, this will be of the form $o(\delta)$.

Conclusion: The limit process obeys $dY_t = \alpha Y_t dt + \gamma Y_t^2 dW_t$.