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Econ 610 Final Exam: Stochastic Dynamic Optimization

Closed book exam. Have fun. Think clearly and correctly formulate your problem first. Then ... Be brief, rigorous, and neat. Your dear friend Katya has checked the exam for clarity and correctness and suggested the following two lines be included:

- If $x \sim N(\mu, \sigma^2)$, its density is given by $f(x) = \exp(-(x-\mu)^2/2\sigma^2)/(\sqrt{2\pi}\sigma)$
- *A* notational reminder: the flow variance of W_t is 1.

1. [(Going into) Labour] Suppose that you wish to have a baby, live in discrete time $0, 1, 2, \ldots$, and are one of the lucky infinitely-lived individuals. You are impatient, with discount factor $\delta \in (0,1)$ on future payoffs. To have a chance $\alpha \in (0,1)$ of having a baby, you must pay an effort cost α^2, and the baby (if any) only arrives *next period.* For each and every birth, you get a one-shot *marginal* benefit of $B > 0$, which is constant in the number of babies. Solve for the optimal α, and your optimal value. [5]

2. What is *probability density* of the following *joint* event: a Wiener process starting at 0 goes through level $W = 5$ at time 5, *and* then through $W = 1$ at time 7. [10]

3. (a) What is the (flow) variance of W_t/e^{W_t} at time t? [5]

   (b) Solve the stochastic differential equation $dX_t = (X_t/t)dt + t dW_t$. [8]

4. You are betting on either a tortoise stock or a hare in a race. The payoff to betting R roubles is $(2R - T)$ roubles, where T is the time it takes your favourite animal to rise from $x = 0$ (the common starting place) to $x = 1$. Each animal's location at time t is given by X_t, which obeys a diffusion of the form $dX_t = \mu dt + \sigma dW_t$. The tortoise has a high drift μ, but low variance σ. The hare has the reverse.

   (a) Find (with non-hand-waving proof) the probability $p(x)$ that that the tortoise rises from x to 1 *in some finite time*, assuming he has a positive drift $\mu > 0$. Hint: Use the fact that $0 \leq p(x) \leq 1$ for *all* x. [10]

   (b) [Bonus] Repeat part (a) for $\mu < 0$. [10]

   (c) Suppose you are condemned to bet $R = 50$ roubles (no more, no less). Which animal would you bet on? (Pretend you are risk neutral and that 100 roubles is worth something to you). [12]

   Hint: You may wish to use part (a) to justify any steps you take here.

5. A diffusion process starts at $x \in (0,1)$. You wish to *minimize the expected time* it takes to hit 0 or 1. The law of motion of the process is $dX_t = u_t dW_t$, where $u_t \in [0,1]$ is a time t *control. Precisely* characterize the optimal behavior and value for any linear flow cost function $c(u) = \alpha u$, where $\alpha > 0$. [14]

6. [I.O. (or the Thomas Edison problem, in continuous time]

(a) Consider an optimal research effort allocation problem. A risky project is characterized by a diffusion net flow payoff $dX_t = \mu dt + \sigma dW_t$, with the current level x of the process. Flow payoffs for the project are discounted by the positive interest rate $r > \mu$. At any time, you can abandon the project and switch to a safe one with a constant flow payoff of $r\gamma$ (that is, at any time, the value of the fall-back alternative is γ). What is your optimal strategy? Graph the optimal value as a function of x, respecting (and commenting on) any monotonicity and convexity/concavity. [20]

Hint: Eliminate one root by arguing that a solution cannot blow up too fast.

(b) Assume there are finitely many projects, each characterized by a diffusion flow payoff $dX_t^i = \mu^i dt + \sigma^i dW_t$ and current level x^i of the process. Flow payoffs for whatever project is operated are discounted by the positive interest rate $r > \mu^i$ for all i. Only one project may be operated at a time. When you stop operating a project, its state X_t^i freezes, remaining stationary until you restart it. Precisely calibrate each project by an index $\gamma(x, \mu, \sigma)$ such that one always operates the project with the highest current index. In particular, show that if two projects have the same current level of the process x, the one with the higher $(\mu + \sqrt{\mu^2 + 2r\sigma^2})/\sigma^2$ will be operated. [16]

Hint: This is largely conceptual, given what we have learned; if you understand the logic, it's a near gift.