HW9: Solutions.

Prob. 1. Prove the Holstein – Primakoff Transition

$$\begin{cases} S_i^+ = a_i^+ \sqrt{2S - a_i^+ a_i} \\ S_i^- = \sqrt{2S - a_i^+ a_i}\, a_i \\ S_i^z = a_i^+ a_i - S \end{cases}$$

Proof: Take the commutator

$$[S_i^+, S_i^-] = [a_i^+ \sqrt{2S - a_i^+ a_i},\ \sqrt{2S - a_i^+ a_i}\, a_i]$$

$$= a_i^+(2S - a_i^+ a_i) a_i - \sqrt{2S - a_i^+ a_i}\, \underbrace{a_i a_i^+}_{a_i^+ a_i + 1} \sqrt{2S - a_i^+ a_i}$$

$$= a_i^+ a_i (2S - a_i^+ a_i) - a_i^+(-1) a_i - (a_i^+ a_i)(2S - a_i^+ a_i) - (2S - a_i^+ a_i)$$

$$= 2(a_i^+ a_i - S) = 2 S_i^z$$

$$[S_i^+, S_i^z] = [a_i^+ \sqrt{2S - a_i^+ a_i},\ a_i^+ a_i - S]$$

$$= a_i^+ \sqrt{2S - a_i^+ a_i}\,(a_i^+ a_i - S) - (a_i^+ a_i - S) a_i^+ \sqrt{2S - a_i^+ a_i}$$

$$= a_i^+ \sqrt{2S - a_i^+ a_i}\,(a_i^+ a_i - S) - a_i^+ \sqrt{2S - a_i^+ a_i}\,(a_i^+ a_i - S) - [a_i^+ a_i, a_i^+] \sqrt{2S - a_i^+ a_i}$$

$$= - a_i^+ \sqrt{2S - a_i^+ a_i} = - S_i^+$$

Similary $[S_i^-, S_i^z] = S_i^-$

so $S_i^z, S_i^\pm$ satisfy the standard commutators for spin- operators

Prob. 2. Diagonize $H = t \sum_{j=1}^{N} (C_j C_{j+1}^{+} + h.c.)$ $(N+1 \equiv 1)$

with a Fourier Transform $C_k = \frac{1}{\sqrt{N}} \sum_j C_j e^{i \frac{2\pi}{N} k j}$

Solution: The inverse Fourier Transform.

$$C_j = \frac{1}{\sqrt{N}} \sum_k C_k e^{-i \frac{2\pi}{N} k j}$$

Substitute it into H

$$H = t \sum_{j=1}^{N} \sum_{k,k'} \frac{1}{N} \left(C_k C_{k'}^{+} e^{-i \frac{2\pi}{N} [kj - k'(j+1)]} + h.c. \right)$$

↑ take summation over j first!

$$= t \sum_{k,k'} \left\{ \underbrace{\left[\frac{1}{N} \sum_j e^{-i \frac{2\pi}{N} j (k-k')} \right]}_{= \delta_{kk'}} C_k C_{k'}^{+} e^{i \frac{2\pi k'}{N}} + h.c. \right\}$$

$$= t \sum_k \left(C_k C_k^{+} e^{i \frac{2\pi k}{N}} + h.c. \right)$$

$$= t \sum_k \left[(1 - C_k^{+} C_k) \cdot 2 \cos \frac{2\pi k}{N} \right]$$

$$= \sum_k \epsilon_k (1 - C_k^{+} C_k)$$

where $\epsilon_k = 2t \cos \frac{2\pi k}{N}$