HW 8 Solutions

It can be readily seen that $\langle p_x \rangle = 0$, thus we have

$$(\Delta x)^2(\Delta p_x)^2 = \frac{1}{9}[\pi^2/3 - \tfrac{1}{2}]\hbar^2 = \hbar^2/4,$$

which is consistent with the Heisenberg uncertainty relation.

Sakurai 6,9,

6. (a) The full wave function for $r>a$ can be written in partial wave analysis as

$$\langle \vec{x}|\psi^{(+)}\rangle = \frac{1}{(2\pi)^{3/2}} \sum_\ell i^\ell(2\ell+1)A_\ell(r)P_\ell(\cos\theta)$$

with $A_\ell = c_\ell^{(1)}h_\ell^{(1)}(kr) + c_\ell^{(2)}h_\ell^{(2)}(kr)$ where $h_\ell^{(1)}$ and $h_\ell^{(2)}$ are the Hankel functions of the first and second kind, respectively. When we consider large r behavior, we have (c.f. (7.6.33)):

$$A_\ell(r) = e^{i\delta_\ell}[j_\ell(kr)\cos\delta_\ell - n_\ell(kr)\sin\delta_\ell].$$

Asymptotically $h_\ell^{(1)} \to e^{i(kr-\ell\pi/2)}/ikr$, $h_\ell^{(2)} \to e^{-i(kr-\ell\pi/2)}/ikr$, while the large r behavior of $\langle \vec{x}|\psi^{(+)}\rangle$ is (from (7.6.8) and (7.6.16))

$$\langle \vec{x}|\psi^{(+)}\rangle \to \frac{1}{(2\pi)^{3/2}} \sum_\ell (2\ell+1)[e^{2i\delta_\ell}e^{ikr}/2ikr - e^{-i(kr-\ell\pi)}/2ikr]P_\ell(\cos\theta).$$

So clearly $c_\ell^{(1)} = \tfrac{1}{2}e^{2i\delta_\ell}$ and $c^{(2)} = \tfrac{1}{2}$. Thus

$$A_\ell(r) = e^{i\delta_\ell}[\cos\delta_\ell j_\ell(kr) - \sin\delta_\ell n_\ell(kr)].$$

For hard sphere, the boundary condition at $r=a$ is $A_\ell(r)|_{r=a} = 0$ because the sphere is impenetrable. This means $j_\ell(ka)\cos\delta_\ell - n_\ell(ka)\sin\delta_\ell = 0$ or $\tan\delta_\ell = j_\ell(ka)/n_\ell(ka)$. For $\ell=0$ $\tan\delta_0 = \frac{\sin(ka)/ka}{-\cos(ka)/ka} = -\tan(ka)$ or $\delta_0 = -ka$.

(b) We have $f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\, P_\ell(\cos\theta)$ and in the limit when $k \to 0$, the $\ell=0$ partial wave dominates the scattering. Thus $f(\theta) = \frac{1}{k}e^{-ika}\sin(ka)$, and knowing that $d\sigma/d\Omega = |f(\theta)|^2$ we have for the total cross section $\sigma = \int(\frac{d\sigma}{d\Omega})\cdot d\Omega = \int|f(\theta)|^2 d\Omega = \int\frac{1}{k^2}\sin^2(ka)d\Omega = 4\pi\sin^2(ka)/k^2 \approx 4\pi a^2$.

Classically, the "geometric cross section" is πa^2. By "geometric cross section" we mean the area of the disc of radius a that blocks the propagation of the plane wave (and has the same cross section area as that of a hard sphere). Low energy scattering of course means a very large wave length scattering and we do not necessarily expect a classically reasonable result.

7. (a) For the Gaussian potential (c.f. (7.4.14)), we have $\Delta_G(b) \equiv \frac{-m}{2k\hbar^2}\int_{-\infty}^{\infty} V(\sqrt{b^2+z^2})dz$ where $V(r) = V_o e^{-r^2/a^2}$. This implies that

$$\Delta_G(b) = \frac{-mV_o}{2k\hbar^2}\int_{-\infty}^{\infty} e^{-(b^2+z^2)/a^2}dz$$

$$= \frac{-mV_o}{2k\hbar^2} e^{-(b/a)^2}\int_{-\infty}^{\infty} e^{-(z/a)^2}dz = -\frac{\sqrt{\pi}}{2}\frac{mV_o a}{k\hbar^2} e^{-(b/a)^2}.$$

Since we are given that $\delta_\ell^G = \Delta(b)|_{b=\ell/k}$, hence

$$\delta_\ell^G = -\frac{\sqrt{\pi}}{2}\frac{mV_o a}{k\hbar^2} e^{-(\ell/ka)^2}$$

(b) For the Yukawa potential $V(r) = V_o e^{-\mu r}/\mu r$, we have

$$\Delta_Y(b) = -\frac{mV_o}{2k\hbar^2}\int_{-\infty}^{\infty}\frac{1}{\mu r}e^{-\mu r}\Big|_{r=\sqrt{b^2+z^2}}dz = -\frac{mV_o}{2k\hbar^2}\int_{-\infty}^{\infty}\frac{e^{-\mu\sqrt{b^2+z^2}}}{\mu(b^2+z^2)^{1/2}}dz.$$

The integral (remembering $r^2 = b^2+z^2$)

$$I = \int_{-\infty}^{\infty}\frac{e^{-\mu(b^2+z^2)^{1/2}}}{(b^2+z^2)^{1/2}}dz = 2\int_0^{\infty}\frac{e^{-\mu(b^2+z^2)^{1/2}}}{(b^2+z^2)^{1/2}}dz = 2\int_0^{\infty}\frac{e^{-\mu r}}{r}\frac{r dr}{(r^2-b^2)^{1/2}}$$

$$= 2\int_0^{\infty}\frac{e^{-\mu r}dr}{(r^2-b^2)^{1/2}} = 2K_o(\mu b)$$

where K_o is the modified Bessel function. Thus

$$\Delta_Y(b) = -\frac{mV_o}{2k\hbar^2}\frac{2K_o(\mu b)}{\mu} = -\frac{mV_o}{\mu k\hbar^2}K_o(\mu b)$$

hence $\delta_\ell^Y = \Delta_Y(b)|_{b=\ell/k}$ assumes value

$$\delta_\ell^Y = -\frac{mV_o}{\mu k\hbar^2}K_o(\mu\ell/k).$$

In case of Gaussian potential $\delta_\ell^G \propto e^{-(\ell/ka)^2}$, and as ℓ increases $\delta_\ell^G \to 0$ very [c.f. (7.6.8)] as

$$A_\ell(k;r) \xrightarrow[r\to\infty]{} \frac{i^{-\ell}}{2ik}\{[1+2ikf_\ell(k)]\frac{e^{ikr}}{r} - \frac{e^{-i(kr-\ell\pi)}}{r}\}. \quad (15)$$

Comparing (14) and (15) and noting (c.f. (7.6.14) and (7.6.15)) that $S_\ell \equiv e^{2i\delta_\ell} = 1 + 2ikf_\ell(k)$, we have

$$f_\ell(k) = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = -(2m/\hbar^2)\int_0^\infty j_\ell(kr)A_\ell(k;r)V(r)r^2dr. \quad (16)$$

9. (a) From (7.6.29), the scattering wave is $\langle\vec{x}|\psi^{(+)}\rangle = \frac{1}{(2\pi)^{3/2}}\sum_\ell i^\ell(2\ell+1)A_\ell(r) \times P_\ell(\cos\theta)$ and $A_\ell(r)$ satisfies (c.f. (7.6.36)) $u_\ell'' + (k^2 - \frac{2mV(r)}{\hbar^2} - \frac{\ell(\ell+1)}{r^2})u_\ell(r) = 0$ with $u_\ell(r) = rA_\ell$. For S-wave and $(2m/\hbar^2)V(r) = \gamma\delta(r-R)$, we consider $\ell=0$ only. Hence $u_o'' + (k^2-\gamma\delta(r-R))u_o = 0$ and for $r<R$ the solution can be written as $u_o(r) = B_o r\sin kr/kr$ while for $r>R$ (using (7.6.33) and (7.6.45)) we have $u_o(r) = re^{i\delta_o}\sin(kr+\delta_o)/kr$. These two solutions must match at $r=R$, i.e. $u_o|_{r=R+} = u_o|_{r=R-} = u_o(R)$ while $u_o'|_{r=R+} - u_o'|_{r=R-} = \gamma u_o(R)$. Therefore

$$\frac{Re^{i\delta_o}\sin(kR+\delta_o)}{kR} = \frac{B_oR\sin(kR)}{kR} \quad (1)$$

$$e^{i\delta_o}\cos(kR+\delta_o) - B_o\cos(kR) = \frac{\gamma B_o}{k}\sin(kR).$$

Solving (1) for $\tan\delta_o$, we have

$$\tan\delta_o = \frac{(-\gamma/k)\sin^2(kR)}{1+(\gamma/k)\sin(kR)\cos(kR)}. \quad (2)$$

(b) Assume $\gamma \gg 1/R$, k, from (1) we obtain

$$\tan(kR+\delta_o) = \frac{\sin(kR)}{\cos(kR)+(\gamma/k)\sin(kR)} \approx \frac{\tan(kR)}{(\gamma/k)\tan(kR)} = \frac{k}{\gamma} \ll 1, \quad (3)$$

thus $-kR \approx \delta_o$, and this resembles the hard sphere scattering (7.6.44). Again from (2) above we have

$$\cot\delta_o = 0 \text{ when } 1 + (\gamma/k)\sin(kR)\cos(kR) = 0 \quad (4)$$

i.e. $\sin(2kR) = -2k/\gamma \approx 0$. Ostensibly we have solutions $(kR)_r \approx n\pi$, $(n+\frac{1}{2})\pi$, but $(n+\frac{1}{2})\pi$ is eliminated since $\cot\delta_o$ then goes through zero from below (negative side). So we write $k_rR \approx n\pi$ (where $d\cot\delta_o/dk < 0$ as k increases) and $k_rR = n\pi - \varepsilon$, $\varepsilon \ll 1$. Hence $\sin(2k_rR) = -\sin(2\varepsilon) = -2k/\gamma$, and $\varepsilon \approx k/\gamma$ to first order, and $k_rR = n\pi - k/\gamma$ as the resonance condition. The resonance energy is

$$E_r = \hbar^2k_r^2/2m \approx \frac{\hbar^2}{2m}\frac{n^2\pi^2}{R^2}(1 - 2/R\gamma). \tag{5}$$

For a particle confined inside potential $V = 0$, $r<R$, and $V = \infty$ for $r>R$ and in S-wave, we have $u'' + k^2u = 0$ where $u(0) = 0$ and $u(R) = 0$. Solution is $u(r) = A\sin(kr)$ $(0\leq r\leq R)$ and from boundary condition $kR = n\pi$, bound state energies are $E_b = \hbar^2k^2/2m = \hbar^2n^2\pi^2/2mR^2$. Hence from (5), we have

$$E_r \approx E_b(1-2/R\gamma). \tag{6}$$

Finally from (2), we have

$$d(\cot\delta_o)/dE = (d(\cot\delta_o)/dk)(dk/dE)$$
$$= \frac{1}{\gamma\sin^4(kR)}[R\cos(kR)(k+\gamma\sin(2kR)/2)\sin(kR) - (1+\gamma R\cos(kR))\sin^2(kR)]\frac{m}{\hbar^2k}. \tag{7}$$

At $E=E_r$, since $k_rR \approx n\pi(1-\frac{1}{\gamma R})$, $\sin^2(k_rR) \approx (n\pi/\gamma R)^2$ and $\cos(2k_rR) \approx 1$, we have from (7)

$$\Gamma = -2/[d(\cot\delta_o)/dE]|_{E=E_r} \approx \frac{2\hbar^2}{m}\frac{(n\pi)^3}{\gamma^2R^4}. \tag{8}$$

Notice that because of the $1/\gamma^2$ dependence in (8), $\Gamma \to 0$ as γ becomes large, thus the resonances become extremely sharp.

10. Assume that initially (t=0) the particle is in an eigenstate $|i\rangle$. The potential $V(\vec{r},t) = V(\vec{r})\cos\omega t$ is turned on at t=0. Take the perturbation expansion of the state amplitudes $c_n(t)$ up to first order $c_n(t) = c_n^{(o)}(t) + c_n^{(1)}(t) + \ldots\ldots$ Then obviously $c_n^{(o)}(t) = \delta_{ni}$. Let the final state be $|f\rangle$, then

$$c_f^{(1)}(t) = (-i/\hbar)\int_0^t V_{fi}(\vec{r})\cos\omega t' e^{i\omega_{fi}t'}dt', \tag{1}$$