HW7: Solutions Albers.

Chapter 8

Potential Scattering

Problem 8.1 Scattering by an Atom

Part a) Let $\mathbf{r}$ and $\mathbf{r}_2$ be the coordinates of the scattering electron and the bound electron respectively. Then in the Born approximation

$$f(\theta,\phi) = \frac{1}{4\pi} 2me^2 \int e^{i\mathbf{q}\cdot\mathbf{r}} \left(\frac{1}{r} - \int |\psi_{100}(r_2)|^2 \frac{1}{|\mathbf{r}-\mathbf{r}_2|} d^3r_2 \right) d^3r \tag{S8.1}$$

where q is the momentum transfer. Of course the potential in the second term can be computed using the fact that its contribution to $V(r)$ is the potential of that part of the bound-electron charge that is inside the sphere $r_2 = r$. Resist the temptation to compute the second term that way. Instead make a change of integration variables in the second term: For fixed $\mathbf{r}_2$ set $\mathbf{r}_1 = \mathbf{r} - \mathbf{r}_2$. Then

$$f(\theta,\phi) = \frac{1}{4\pi} 2me^2 \int \left(e^{i\mathbf{q}\cdot\mathbf{r}_1} \frac{1}{r_1} - \int |\psi_{100}(r_2)|^2 e^{i\mathbf{q}\cdot\mathbf{r}_1} e^{i\mathbf{q}\cdot\mathbf{r}_2} \frac{1}{r_1} d^3r_2 \right) d^3r_1 \tag{S8.2}$$

In the first term I replaced $\mathbf{r}$ by $\mathbf{r}_1$ as the integration variable.[1] Thus (change $\mathbf{r}_1$ back to $\mathbf{r}$)

$$f(\theta,\phi) = \frac{1}{4\pi} 2me^2 \int e^{i(\mathbf{q}\cdot\mathbf{r}+i\epsilon r)} \frac{1}{r} \left(1 - \int |\psi_{100}(r_2)|^2 e^{i\mathbf{q}\cdot\mathbf{r}_2} d^3r_2 \right) d^3r \tag{S8.3}$$

The double integral factors! The first factor is an integral familiar from the pure Coulomb case:

$$\int e^{i(\mathbf{q}\cdot\mathbf{r}+i\epsilon r)} \frac{1}{r} d^3r = \frac{4\pi}{q^2+\epsilon^2} \to \frac{4\pi}{q^2} \tag{S8.4}$$

In the second factor use the explicit form of the ground state wave function:

$$\int |\psi_{100}(r)|^2 e^{i\mathbf{q}\cdot\mathbf{r}} d^3r = \frac{1}{\pi a^3} \int e^{i\mathbf{q}\cdot\mathbf{r}} e^{-2r/a} d^3r = \frac{16}{a^4} \left[\frac{1}{q^2 + 4/a^2} \right]^2 \tag{S8.5}$$

$$\boxed{f(\theta,\phi) = \frac{1}{4\pi} 2me^2 \frac{4\pi}{q^2} \left(1 - \frac{16}{(q^2a^2+4)^2} \right) = 2a \frac{q^2a^2+8}{(q^2a^2+4)^2}} \tag{S8.6}$$

The scattering amplitude $f(\theta,\phi)$ approaches Rutherford's formula as $q^2 \to \infty$, but is finite as $q^2 \to 0$. The shielding removes the infinity in the forward direction.

The scattering length is $-a$ (the Bohr radius). The threshold cross section is $4\pi a^2$.

[1] Since the first integral is not unambiguously convergent, the factoring is slightly illegal. But we could have replaced $1/r$ by $e^{-\epsilon r}/r$ and taken the limit $\epsilon \to 0$ at the end.

Part b) For arbitrary k and θ, the differential cross section is

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 = 4a^2(q^2a^2+8)^2\frac{1}{(q^2a^2+4)^4}} \tag{S8.7}$$

Again, for large q^2 this becomes the Rutherford cross section.

The total cross section is

$$\sigma = 8\pi a^2\int_{-1}^{1}(q^2a^2+8)^2\frac{1}{(q^2a^2+4)^4}d(\cos\theta) = \frac{4\pi}{k^2}\int_0^{4k^2a^2}(q^2a^2+8)^2\frac{1}{(q^2a^2+4)^4}d(q^2a^2) \tag{S8.8}$$

Set $\lambda = k^2a^2$. The integral (with $y = q^2a^2+4$) is

$$\int_0^{4(\lambda+1)}\frac{(y+4)^2}{y^4}dy = \left.\frac{y^2+4y+16/3}{y^3}\right|_4^{4(1+\lambda)} = \frac{\lambda}{4(1+\lambda)^3}\left(4+6\lambda+\frac{7}{3}\lambda^2\right) \tag{S8.9}$$

The total cross section is

$$\boxed{\sigma = \pi a^2\frac{1}{(1+k^2a^2)^3}\left(4+6k^2a^2+\frac{7}{3}k^4a^4\right)} \tag{S8.10}$$

Problem 8.2 Scattering by an Attractive Square Well

Part a) Let $u_l = rR_l$ as usual. Because the partial wave amplitudes go like k^{2l+1} as $k \to 0$, only the $l = 0$ state is important near zero energy. Inside the well, u_o is proportional to $\sin(k_1r)$, so

$$\frac{u_o'}{u_o} = k_1\cot(k_1r) \tag{S8.11}$$

Outside the well, u_o is proportional to

$$\sin(kr+\delta_o) = \sin(kr)\cos\delta_o + \cos(kr)\sin\delta_o \tag{S8.12}$$

Both $u_o(r)$ and its derivative are continuous at $r = a$. Equating the logarithmic derivative of this form to (S8.11) provides the equation for the phase shift (analogous to the eigenvalue equation in the bound state problem).

$$\frac{k\cos(ka)\cos\delta_o - k\sin(ka)\sin\delta_o}{\sin(ka)\cos\delta_o + \cos(ka)\sin\delta_o} \tag{S8.13}$$

or

$$\cot\delta_o = \frac{k\sin(ka) + k_1\cot(k_1a)\cos(ka)}{k\cos(ka) - k_1\cot(k_1a)\sin(ka)} \tag{S8.14}$$

As $k \to 0$ the partial wave expansion (8.93) becomes

$$f(\theta,\phi) \to \frac{\delta_o}{k} + \cdots \tag{S8.15}$$

The scattering length is

$$-\lim_{k\to 0} f(\theta,\phi) \approx -\lim_{k\to 0}\frac{\delta_o}{k} \approx -\lim_{k\to 0}\frac{1}{k\cot\delta_o} = a - \lim_{k\to 0}\frac{\tan(k_1a)}{k_1} = \boxed{a - \frac{\tan(k_oa)}{k_o}} \tag{S8.16}$$

The threshold total cross section is

$$\sigma \to 4\pi|f(\theta,\phi)|^2 = 4\pi\left|a - \frac{\tan(k_oa)}{k_o}\right|^2 \tag{S8.17}$$

Part b) Use the rule (8.40) for the Born approximation to the scattering amplitude when the potential is spherically symmetric, with $U(r) = 2mV(r) = -2mV_o\Theta(a-r)$:

$$f(\theta,\phi) = -\frac{1}{q}\int_0^\infty r\sin(qr)U(r)dr = \frac{2mV_o}{q}\int_o^a r\sin(qr)dr = \boxed{\frac{2mV_o}{q^3}\left[\sin(qa) - qa\cos(qa)\right]} \tag{S8.18}$$

As $k \to 0$, $q \to 0$ also, and

$$\sin(qa) - qa\cos(qa) \to \frac{(qa)^3}{3} \tag{S8.19}$$

so

$$f(\theta,\phi) \to \frac{2mV_o a^3}{3} = \frac{a}{3}(k_o a)^2 \tag{S8.20}$$

For small k_o this is the same as the limit of the exact $l = 0$ scattering amplitude in (S8.16). But when k_o becomes comparable to $2mV_o/a^2$, the Born approximation ceases to be good. That is the region where the bound states begin to appear.

Problem 8.3 A Neutron-Proton Scattering Model—I
Part a) The scattering length is

$$A = a - \frac{\tan(k_o a)}{k_o} \tag{S8.21}$$

Here $a = 1/m_\pi$ and $k_o^2 = 2\mu V_o = m_N V_o$. Take $V_o = 66.3$MeV. Then

$$k_o a = \frac{\sqrt{m_N V_o}}{m_\pi} = \frac{\sqrt{940 \times 66.3}}{140} \tag{S8.22}$$

and

$$A = \left(1 - \frac{\tan(k_o a)}{k_o a}\right) a = 5.06 \times 10^{-13}\text{cm} \tag{S8.23}$$

The threshold cross section is substantially smaller than the experimental result:

$$\boxed{\sigma = 3.198 \times 10^{-24}\text{cm}^2} \tag{S8.24}$$

Part b) Table S8.1 shows the potential strength V_o needed to get the correct binding energy, and then the threshold cross section for the resulting potential, for a range of ranges a.

a (cm)	V_o (MeV)	σ (cm^2)
1.409×10^{-14}	5282.0	2.422×10^{-24}
2.819×10^{-14}	1356.0	2.501×10^{-24}
7.047×10^{-14}	234.3	2.749×10^{-24}
1.409×10^{-13}	66.42	3.198×10^{-24}
2.819×10^{-13}	21.07	4.230×10^{-24}
7.047×10^{-13}	6.284	8.292×10^{-24}
1.409×10^{-12}	3.513	1.054×10^{-23}

Table S8.1: V_o and σ for a range of a in the spherical square well model

There is no value of a within this wide range that comes close to giving the observed value of σ.

Part c) There are bound states when $V_o \geq \pi^2/8ma^2$ (Problem 3.11). From Problem 8.2, the scattering length is $A = (k_o a - \tan k_o a)/k_o$ where $k_o^2 = 2mV_o$. So when V_o has the minimum value, $k_o a = \pi/2$, and $\tan k_o a = \infty$.