HW4: Solutions

so here

$$N = \left(\frac{4a^6}{\pi}\right)^{\frac{1}{4}} \qquad \langle\psi| T |\psi\rangle = \frac{3\omega}{4} = \frac{3}{4}\frac{a^2}{m} = \frac{3}{2}a^2 \tag{S7.178}$$

$$\langle\psi| V |\psi\rangle = N^2 \int_{-\infty}^{\infty} x^6 e^{-a^2x^2} dx = \left(\frac{4a^6}{\pi}\right)^{\frac{1}{2}} \frac{1}{a^7}\Gamma\left(\frac{7}{2}\right) = \frac{15}{4a^4} \tag{S7.179}$$

The best value of the parameter satisfies

$$0 = \frac{d\bar{E}}{da^2} = \frac{3}{2} - \frac{15}{2a^6} \tag{S7.180}$$

or $a^6 = 5$. The energy estimate is

$$\boxed{\bar{E} = \frac{9}{4}5^{\frac{1}{3}} = 3.847\ldots} \tag{S7.181}$$

This too is greater than the true value, by about 1 %.

Problem 7.15 WKB Estimate for a Quartic Potential

The phase integral in equation (7.252) is

$$\Phi = \int_{x_-}^{x_+} \sqrt{2m\left(E - \lambda x^4\right)}dx = 2\int_0^{x_+} \sqrt{2m\left(E - \lambda x^4\right)}dx \tag{S7.182}$$

where the turning points $x_\pm$ are solutions to $E = \lambda x^4$:

$$x_\pm = \pm\left(\frac{E}{\lambda}\right)^{\frac{1}{4}} \tag{S7.183}$$

With a simple change of integration variable ϕ becomes

$$\Phi = 2\left(\frac{4m^2}{\lambda}\right)^{\frac{1}{4}} E^{\frac{3}{4}} I \tag{S7.184}$$

where

$$I = \int_0^1 \left(1 - t^4\right)^{\frac{1}{2}} dt \approx 0.8740196081 \tag{S7.185}$$

Then from equation (7.255)

$$E = \bar{E}\left(\frac{\lambda}{4m^2}\right)^{\frac{1}{3}} \tag{S7.186}$$

where

$$\bar{E} = \left(\frac{\pi}{2I}\right)^{\frac{4}{3}} \left(n + \frac{1}{2}\right)^{\frac{4}{3}} \approx 2.1850678891 \left(n + \frac{1}{2}\right)^{\frac{4}{3}} \tag{S7.187}$$

The lowest values are

$$\begin{aligned} \bar{E}_0 &= 0.8671447665 \\ \bar{E}_1 &= 3.7519175005 \\ \bar{E}_2 &= 7.4139834646 \end{aligned} \tag{S7.188}$$

The fractional errors are about 18.2%, 1.3%, and 0.56%, respectively. As expected, the WKB approximation improves rapidly for the higher states.

terms and corresponding higher order excited states.

24. The initial state is $|0\rangle$, so from (5.6.17), we have

$$c_n^{(o)}(t) = \delta_{no},\ c_n^{(1)}(t) = (-i/\hbar)\int_0^t e^{-i(E_o-E_n)t'/\hbar}\langle n|H'(x,t')|0\rangle dt'. \quad (1)$$

Next we note that

$$\langle n|H'(x,t)|0\rangle = Ae^{-t/\tau}\langle n|x^2|0\rangle \quad (2)$$

and from (2.3.24), we have $x^2|0\rangle = \frac{1}{2}(\hbar/m\omega)(a+a^\dagger)(a+a^\dagger)|0\rangle$. Since $a|0\rangle = 0$, $a^\dagger|0\rangle = |1\rangle$, $a|1\rangle = |0\rangle$, $a^\dagger|1\rangle = \sqrt{2}|2\rangle$, thus $x^2|0\rangle = (\hbar/2m\omega)[|0\rangle + \sqrt{2}|2\rangle]$, and therefore $\langle n|x^2|0\rangle = (\hbar/2m\omega)[\delta_{no}+\sqrt{2}\delta_{n2}]$. We see that if $n \neq 0$ or $n \neq 2$, $c_n^{(1)}(t)$ of (1) vanishes because $\langle n|x^2|0\rangle$ vanishes in (2). Only the following coefficients are relevant to our discussion: $c_o^{(o)}=1$, $c_2^{(o)}=0$, $c_o^{(1)} = (-i/\hbar)\int_0^t(\hbar/2m\omega)\times Ae^{-t'/\tau}dt' = \frac{iA}{2m\omega}(e^{-t/\tau}-1)\tau$ (which for $t/\tau \gg 1$, gives $c_o^{(1)} \simeq -iA\tau/2m\omega$), $c_2^{(1)} = (-i/\hbar)\frac{\hbar}{2m\omega}\sqrt{2}\int_0^t \exp[-i(E_o-E_2)t'/\hbar]Ae^{-t'/\tau}dt' = -i\sqrt{2}A/2m\omega(1/\tau - 2\omega i)$.

After a long time duration of perturbation, the state becomes [see (5.5.4) and (5.6.1)]

$$|\psi\rangle = [1 - iA\tau/2m\omega]e^{-i\omega t/2}|0\rangle - \frac{i\sqrt{2}A}{2m\omega(1/\tau-2i\omega)}e^{-i5\omega t/2}|2\rangle \quad (3)$$

(Remark: higher order terms like A^2, A^3,...... are ignored.) So the probability for the system to be transmitted to the second excited state is

$$P_2 = \frac{|A|^2}{2m^2\omega^2(1/\tau^2+4\omega^2)}\bigg/[1+\frac{|A|^2\tau^2}{4m^2\omega^2} + \frac{1}{2}\frac{|A|^2}{m^2\omega^2(4\omega^2+1/\tau^2)}]. \quad (4)$$

There is no probability for transition to other states such as $|1\rangle, |3\rangle$,.......

25.

$$H = \begin{pmatrix} E_1^{(o)} & \lambda\cos\omega t \\ \lambda\cos\omega t & E_2^{(o)} \end{pmatrix} = H_o + V(t)$$

(a) Let us write $|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. A general state is

$$|\alpha,t\rangle = c_1(t)\exp[-iE_1^{(o)}t/\hbar]\ |1\rangle + c_2(t)\exp[-iE_2^{(o)}t/\hbar]\ |2\rangle$$

Prob. 3 Solution:

Let $\vec{S} = \frac{1}{2} \sum_i \vec{\sigma}_i$

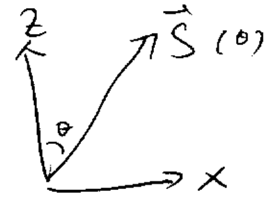
$\vec{S}$ will be a collective spin operator with $\vec{S}^2 = S(S+1)$

$$S = \frac{N}{2} \quad \text{(addition of } N \ \tfrac{1}{2} \text{ spins)}$$

Hamiltonian $H = -4 S_z^2 + 2B S_x$

Under the mean-field approx. $\langle S_z^2 \rangle = \langle S_z \rangle^2$

$$\langle H \rangle = -4 \langle S_z \rangle^2 + 2B \langle S_x \rangle$$



Parametrize $\begin{cases} \langle S_z \rangle = \frac{N}{2} \cos\theta \\ \langle S_x \rangle = \frac{N}{2} \sin\theta \end{cases}$

$$\langle H \rangle = -N^2 \cos^2\theta + BN \sin\theta$$
$$= N^2 \sin^2\theta + BN \sin\theta - N^2$$
$$= N^2 \left(\sin\theta + \frac{B}{2N} \right)^2 - N^2 - \frac{B^2}{4}$$

when $\left| \frac{B}{2N} \right| \le 1$ minimum $\sin\theta = -\frac{B}{2N}$

when $\frac{B}{2N} > 1$ minimum $\sin\theta = -1$

phase Transition point $B = 2N$