\[ f(x,y) = (x-4) \ln(xy) \]

\[ f_x = (x-4) \cdot \frac{1}{x} + \ln(xy) \quad f_y = (x-4) \cdot \frac{1}{y} \cdot x \]

\[ 0 = \frac{x-4}{x} + \ln(xy) \quad 0 = \frac{x-4}{y} \]

\[ 0 = 0 + \ln(4y) \]

\[ e^x = e^{\ln(y)} \]

\[ (x, y) = \left( 4, \frac{1}{4} \right) \]

\[ f_{xx} = \left( \frac{(x)(x)-(x-4)(1)}{x^2} \right) + \frac{1}{xy} \quad f_{yy} = \left( \frac{y^2}{y^2} \right) - \left( \frac{x-4}{y^2} \right) \]

\[ f_{xy} = \frac{1}{xy} \quad x = \frac{1}{y} \]

For \( (x, y) = \left( 4, \frac{1}{4} \right) \)

\[ \left( \frac{4+4}{(4^2)} \right) \left( \frac{-4+4}{(\frac{1}{4})^2} \right) = \left( \frac{1}{\frac{1}{4}} \right) \left( \frac{1}{\frac{1}{4}} \right) \]

\[ 0 = 16 = -16 \]

**Saddle Point**
Now that we know how to minimize and maximize functions of two variables, we can min/max functions given certain restraints.

Example:
If we have a production function:

\[ Q = x^{\frac{1}{3}} y^{\frac{2}{3}} \]

where \( Q \) = output, \( x \) = labor, \( y \) = capital

and the cost of \( x = 2 \) and \( y = 6 \) - what is the minimum amount of output you can achieve minimum cost you could spend on producing 1000 units?

What is the thing we are looking to optimize? \( \rightarrow \) Cost

\[ f = 2x + 6y \]

TOTAL COST

Subject to what constraint?

\[ Q = x^{\frac{1}{3}} y^{\frac{2}{3}} = 1000 \]

OUTPUT MUST EQUAL 1000.

The method of Lagrange multipliers helps us to solve this equation, finding the values of \( x \) and \( y \) that optimize while achieving our constraint.

In terms of Lagrange multipliers, we term the function we want to minimize/maximize \( f \) and the constraint function \( g \).
If we picture a function of two variables as a set of level curves

\[ f(x,y) = C \]

\[ g(x,y) = C \]

The point where the constraint curve and the level curve are tangent is the optimization.

As it turns out, the slope of the function \( f \) at any point is \( \frac{-fx}{fy} \).

What do you think the slope of the function \( g \) is at any point?

\[ \frac{-gx}{gy} \]

And at the constrained maximization, what do you think is true about both of these slopes?

\[ \frac{-fx}{fy} = \frac{-gx}{gy} \]
so, \( \frac{f_x}{q_x} = \frac{f_y}{q_y} \) Algebraically.

If we call the ratio between the derivatives something in common (\( \lambda \))

\[
\frac{f_x}{q_x} = \lambda \quad \frac{f_y}{q_y} = \lambda
\]

\[
f_x = (\lambda)q_x \quad f_y = (\lambda)q_y \quad \text{and} \quad g(x, y) = K
\]

(Constraint equation)

So, to solve our previous example:

\( f = 2x + 6y \quad g = x^\frac{1}{3} y^\frac{2}{3} = 1000 \)

\[
\begin{align*}
\frac{f_x}{q_x} &= 2 \\
\frac{f_y}{q_y} &= 6 \\
2 &= \frac{\Delta}{2} x^\frac{1}{3} y^\frac{2}{3} \\
\lambda &= 4x^\frac{1}{3} y^\frac{2}{3} \\
4x^\frac{1}{3} y^\frac{2}{3} &= 18x^\frac{1}{3} y^\frac{2}{3} \\
x &= 4.5y
\end{align*}
\]

(4.5y)^\frac{2}{3} y^\frac{2}{3} = 1000 \quad y = 14.16 \quad \lambda = 4.5

4 \times 168.68 \quad x = \frac{926.8}{4} \quad \text{Cost} = 24,219.40

\[
\begin{align*}
\frac{\sqrt[3]{45} y^\frac{2}{3}}{y^\frac{2}{3}} &= 1000 \\
y^\frac{2}{3} &= 471.4
\end{align*}
\]
Find the maximum value of \( f(x,y) = xy \) subject to the constraint \( x + y = 1 \).

**Optimizing** \( \quad f = xy \)

**Constraint** \( \quad g = x + y = 1 \)

\[
\begin{align*}
\frac{f}{x} &= y \\
\frac{f}{y} &= x \\
\frac{g}{x} &= 1 \\
\frac{g}{y} &= 1
\end{align*}
\]

Find three Lagrange equations:

\[
\begin{align*}
\frac{f}{x} &= \lambda \frac{g}{x} \\
\frac{f}{y} &= \lambda \frac{g}{y} \\
\frac{g}{x} &= 1 \\
\frac{g}{y} &= 1
\end{align*}
\]

If we choose any other combination of \( x \) and \( y \), \( f \) will be less than \( \frac{1}{4} \).

\( \lambda = \frac{1}{2} \) if \( k \) increases by one, how will \( f \) change?

**LAGRANGE STEPS**

1. Identify \( f \) (optimize function) and \( g \) (constraint function).
2. Find \( f_x, f_y, g_x, g_y \).
3. Find three Lagrange equations: \( \begin{align*} 
\frac{f}{x} &= \lambda \frac{g}{x} \\
\frac{f}{y} &= \lambda \frac{g}{y} \\
\frac{g}{x} &= 1 \\
\frac{g}{y} &= 1 
\end{align*} \)
4. Solve the equations simultaneously.
Let's try an example that is a little bit harder...

A farmer wishes to fence off a rectangular pasture along the bank of a river. The area must be 3200 m$^2$ and no fencing along the river is needed. Find the dimensions of the pasture that will require the least amount of fencing.

\[ f = x + 2y \]

\[ g = xy = 3200 \]

\[ f_x = 1 \]
\[ f_y = 2 \]
\[ g_x = y \]
\[ g_y = x \]

1. \( \lambda x = y \)
2. \( \lambda y = x \)
3. \( xy = 3200 \)

\[ \frac{2}{x} = \lambda \]
\[ \frac{2}{y} = \lambda \]

\[ \frac{x}{2} = \lambda \]
\[ \frac{y}{2} = \lambda \]

\[ x \cdot \frac{x}{2} = 3200 \]
\[ \frac{x^2}{2} = 3200 \]
\[ x^2 = 6400 \]
\[ x = 80 \text{ m} \]
\[ \frac{y}{2} = \lambda \]
\[ y = 40 \text{ m} \]
\[ f(\text{min}) = 200 \text{ m} \]
\[ \lambda = \frac{1}{40} \]

If our constraint increased by 1, our \( f \) would increase by 100.