Functions of two variables
Graphical Representation of multivariable functions
Partial derivatives

Move now into calculus of two or more variables:

So far we have only seen functions with one independent variable in these forms:

\[ y = 4x + 2 \]
\[ f(x) = 2x^2 - 1 \]
\[ a(t) = e^{5t} - 4 \]

These one dimensional functions have one input that determines output. These inputs have been things like time → distance or price → profit.

Not everything (as a matter of fact, most things) has only one input but have many inputs. For example, the temperature in this room:

- Air conditioning system temp setting
- Whether the windows are opened or closed
- The temperature outside
- Sunshine into the window
- Number of people in the room.

The temperature in the room could be expressed as a function of all these different factors (or variables).

\[ f(a, b, c, d, e) = a + 4b + \frac{1}{5}c + 20d + \frac{1}{3}e \]
Other multivariate functions include an infinite amount of things. Examples:

Output in terms of capital and labor called the Cobb-Douglas production function. Any sort of linear regression model.

Working with multivariate functions is the same as working with single variable functions.

\[ f(x) = 3x + 2 \]
\[ f(2) = 3(2) + 2 = 8 \]

\[ f(x, y) = 4x + 3y \]
\[ f(2, 1) = 4(2) + 3(1) = 11 \]

The notation is sometimes also similar:

One variable
\[ f(x) \]
\[ 0(x) \]

Multiple Variables
\[ f(x, y) \]
\[ 0(t, u) \]

Like single variable functions, multiple variable functions have domains and ranges.

Recall, the domain is what in inputs can be - and the range is what all the outputs can be.
Examples: \( f(x, y) = 4x - 3y \)

a) Evaluate \( f(x, y) \) at \( f(0, 2) \)
\[ f(0, 2) = 4(0) - 3(2) = -6 \]

b) What is the domain of \( f(x, y) \)?
All real numbers

\[ z = \frac{9x + 3y^2}{x} \]

a) Evaluate \( z \) at \( x = 2 \) and \( y = 1 \)
\[ z = \frac{4(2) + 3(1)^2}{2} = \frac{11}{2} \]

b) What is the domain of \( z \)?
All ordered pairs \( (x, y) \) for \( x \neq 0 \).

\[ f(x, y) = \frac{3x^2 + 5y}{x - y} \]

a) Evaluate \( f(0, -1) \)
\[ f(0, -1) = \frac{3(0)^2 + 5(-1)}{0 - (-1)} = -5 \]

b) Domain: all real number ordered pairs \( (x, y) \) except \( x = y \)

\[ f(a, b) = ae^b + \ln a \]

a) Evaluate \( f(e^2, \ln 2) \)
\[ f(e^2, \ln 2) = ae^{\ln 2} + \ln (e^2) \]
\[ e^{2 \cdot 2} + \ln (e^2) = 2e^2 + 2 = 16.78 \]

b) What is the domain?
Cannot take the natural log of a negative number...
Domain: all real ordered pairs \( (a, b) \) for \( a \neq 0 \).
We will spend much of our time on two variable calculus and moving from there...

**GRAPHICAL REPRESENTATION:**

One independent variable: \( f(x) \) or \( y \)

\[ (4, 6) \]
\[ (2, 1) \]
\[ (-3, 4) \]

Two independent variables: \( f(x, y) \) or \( z \)

\[ (3, 1, 2) \]

With two independent variables, we are dealing with 3 dimensions.

The graphical representation of two variables is difficult — with more variables it is even more difficult to draw. (Imagine drawing four/five/more dimensions).

One relatively simple way to draw an interpretation is to use the Level Curve technique. We have seen this before in things like topographical /elevation maps.
LEVEL CURVES

Topographical / Elevation map example:

We can use the same process for graphing 3-D functions:

\[ f(x,y) = x^2 + y^2 \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>f(x,y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0</td>
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<tr>
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<td>1</td>
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<td>4</td>
<td>0</td>
<td>16</td>
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</tbody>
</table>

Envision a parabola

Does this function exist on any negative levels? So, visually this is like a cone that is coming out of the chalkboard...

What do you think the point (0,0,a) represents? It is a critical point (minimum).

This C is the plane of z (cross-section) also referred to as the isocost in economics, or - Indifference curves.
EXAMPLE: Draw the function \( f(x,y) = x^2 + y \) using level curves

\[ C = 0, \ C = 9, \ C = 18, \ C = 27 \]

This is like a parabola that comes out of the chalkboard.

**MECHANICS OF MULTIPLE VARIABLE FUNCTIONS:**

In the function

\[ f(x,y) = 3x + 4y + 10 \]

what does \( f(x,y) = f(0,0) = 10 \)

\( f(x,y) = f(1,0) = 13 \)

What happened to \( f(x,y) \) when \( x \) went up by one?

\( f(x,y) = f(0,1) = 14 \)

What happened to \( f(x,y) \) when \( y \) goes up by one?

This is just like the definition of slope (how much \( x \) changes... →

\[
\begin{array}{c|cc|cc|c|cc|c}
\hline
\text{C} & \text{X} & \text{Y} & \text{C} & \text{X} & \text{Y} \\
\hline
0 & 0 & 0 & 9 & 0 & 9 \\
0 & 1 & -1 & 9 & 1 & 6 \\
0 & -1 & -1 & 9 & -1 & 6 \\
0 & 2 & -4 & 9 & 2 & 4 \\
0 & -2 & -4 & 9 & -2 & 4 \\
4 & 0 & 4 & 4 & 1 & 3 \\
4 & -1 & 3 & 4 & 2 & 0 \\
4 & -2 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\]
when \( x \) grows up by one or when \( y \) grows up by one?

In effect, the function \( f(x,y) = 3x + 4y \) has two slopes — how \( f(x,y) \) changes with regard to \( x \) and how \( f(x,y) \) changes with regard to \( y \). These are called *partial derivatives*.

\[
\frac{\partial f(x,y)}{\partial x} = 3 \quad \frac{\partial f(x,y)}{\partial y} = 4
\]

What is the change in \( f(x,y) \) with regard to \( x \)? \( \frac{\partial f(x,y)}{\partial x} = 3 \)

What is the change in \( f(x,y) \) with regard to \( y \)? \( \frac{\partial f(x,y)}{\partial y} = 4 \)

In effect, what we are doing is treating the non-relevant variable as a constant.

Notation:

\[
\frac{\partial f(x,y)}{\partial x} = 3 \quad \frac{\partial f(x,y)}{\partial y} = 4
\]

\[
f_x = 3 \quad f_y = 4
\]

So any time we see a function with two or more variables, and we are asked to take the derivative — the first thing we ask is "With respect to what?"

Then — all the other derivative rules we have learned the last four weeks apply.
Write on the board horizontally.

Example:
\[ z = x^2 + 2xy + 4y \]
\[ \frac{dz}{dx} = 2x + 2y \]
\[ \frac{dz}{dy} = 2x + 4 \]
\[ \frac{dz}{dx} - 2 \frac{dz}{dy} = 0 \]
\[ \frac{dz}{dy} - \frac{dz}{dx} = -2 \]

Example:
\[ f(x, y) = 3xy^3 - 2x \]
\[ f_x = 3y^3 - \frac{2}{y} \]
\[ f_y = 6xy + \frac{2x}{y} \]

Example:
\[ z = 4x^2y^2 \]
\[ \frac{dz}{dx} = 12x^2y^2 \]
\[ \frac{dz}{dy} = 8x^3y \]

Example:
\[ z = (x^2 + xy + y^2)^4 \]
\[ \frac{dz}{dx} = 4(x^2 + xy + y^2)^3 (2x + y) \]
\[ \frac{dz}{dy} = 4(x^2 + xy + y^2)^3 (x + 2y) \]

Example:
\[ f(x, y) = xe^{-2xy^2} \]
\[ f_x = (x^2)(e^{-2xy^2})(-2y^2) + (e^{-2xy^2})(2x) \]
\[ f_x = xe^{-2xy^2}(2xy^2 + 2) \]
\[ f_y = (x^2)(e^{-2xy^2})(-4xy) + (e^{-2xy^2})(0) \]
\[ f_y = -4x^3y e^{-2xy^2} \]

How do we find the second derivatives? Just like with the first order partial derivatives, there is an \( x \) and \( y \) derivatives.