We covered so far all the basic applications of derivatives including:
1. Approximation
2. Marginal Analysis
3. Curve Sketching
4. Optimization

What about with advanced functions (such as with $e$ or with natural logs) and with functions of multiple variables? This is what we are going to focus on in Part 2 of this class.

**EXponent Rule Review:**

1. $x^n = x \cdot x \cdot x \cdots \cdot x$  
   \[ \text{Example } x^4 = x \cdot x \cdot x \cdot x \]

2. $x^a \cdot x^b = x^{a+b}$  
   \[ \text{Example } x^3 \cdot x^4 = x^7 \]
   $\frac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x}$  
   \[ \text{Example } \frac{x^4}{x^2} = \frac{x \cdot x \cdot x \cdot x}{x \cdot x} = x^2 \]

   This follows on to the next rule:

3. $x^{-n} = \frac{1}{x^n}$  
   \[ \frac{3}{x^2} = \frac{x^2}{x^2} \text{ by rule 3} \]
   \[ \frac{x \cdot x \cdot x}{x \cdot x \cdot x} = 1 \]

4. $x^0 = 1$  
   \[ \frac{x^3}{x^3} = x^0 \text{ by rule 3} \]

5. $(x^a)^b = x^{a \cdot b}$  
   \[ (x^3)^2 = x^6 \]
   \[ (x \cdot x) \cdot (x \cdot x) \cdot (x \cdot x) = x^5 \]

6. $(x^{\frac{m}{n}}) = m \cdot \sqrt[n]{x^m}$  
   \[ x^{\frac{3}{2}} = \frac{3}{\sqrt[2]{x^3}} \]
Exponential functions typically take the following shapes:

\[ f(x) = b^x \]

for all \( b > 0 \) and \( b \neq 1 \).

A negative base has non-real answers.

A base of \( 1 \) makes \( f(x) = 1 \) for all values.

Zero raised to any power is zero (except the zero power).

**Introducing \( e \)**

Putting money in the bank: No compounding meaning the interest is found once per time period (year).

1 year: \( A + Ar \)

2 years: \( (A + Ar) \cdot (A + Ar) \cdot c \)

3 years: \( (A + Ar)^2 \cdot (A + Ar) \cdot c \)

Non-compounding: \( A(1 + r)^t \) where:
- \( A \) = Principal
- \( r \) = Rate
- \( t \) = Time (years)
Most banks compound interest a set amount of times per year, i.e. quarterly, monthly, semi-annually, etc.

If a bank has a 100% interest rate compounded semi-annually, it will calculate 50% interest at 6 months, and then cumulatively 50% interest in 12 months, meaning you make more money.

$$A + A^{\frac{y}{k}}\frac{1}{t}$$

$k$: times per year calculated

$$A(1 + \frac{y}{k})$$

This is the total money you have after period 1 of $k$.

$$\left(\frac{A + A^{\frac{y}{k}}}{k}\right) + \left(\frac{A + A^{\frac{y}{k}}}{k}\right)^2$$

There are $k$ periods in a year, so the cumulative for a year is $$A(1 + \frac{y}{k})^k$$

For $t$ years, the formula is $$A\left(1 + \frac{y}{k}\right)^{kt}$$

**COMPOUND INTEREST GIVES US:**

$$B = A(1 + \frac{y}{k})^{kt}$$

$B$: total money
$A$: principal
$y$: yearly interest rate
$k$: compound frequency
$t$: number of years compounded.

Without compounding, $k = 1$

$$B = A\left(1 + \frac{y}{k}\right)^{kt}$$

(OFFER FORMULA)
Example:
Suppose $1000 is invested in a bank at the rate of 6% annual interest. Bank A, which does not compound interest, Bank B compounds interest semi-annually, and Bank C compounds interest quarterly. Where should I invest and how much would I make? (If I invested for 3 years)

BANK A: \( B = 1000(1 + \frac{.06}{1})^{1(3)} = 1000(1.06)^3 = 1,191.02 \)

BANK B: \( B = 1000(1 + \frac{.06}{2})^{2(3)} = 1000(1.03)^6 = 1,194.05 \)

BANK C: \( B = 1000(1 + \frac{.06}{4})^{4(3)} = 1000(1.015)^{12} = 1,195.02 \)

ANSWER: the more I compound, the better...

So, it intuitively seems like the more times I compound, the more money I should be making. That is true, but is it a lot of money?

Let's invest one dollar at 100% annual interest for one year.

B = \( 1(1 + .5)^1 = 2 \)

Quarterly: \( B = 1(1 + .25)^4 = 1(1.25)^4 = 2.94 \)

Daily: \( B = 1(1 + \frac{.05}{365})^{365} = 1.00027^{365} \approx 2.718 \)

What about an infinite amount of times? This is known as compounding continuously.

B = \( 1(1 + \frac{.05}{n})^{n\rightarrow\infty} = 2.718 \)

\[ \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \]

This is a practical definition of the number \( e \).
So, a practical definition of the number “e” is that it is what you get when you continuously compound $1 at 100\%$ annual interest for one year. Or, more generally, $e$ is what you get when you start with one unit that increases at a 100\% rate over a time period, compounded continuously.

**Theoretical Basis for “e”**

- $B = A\left(1 + \frac{r}{k}\right)^{kt}$, what happens when we compound continuously as $k \to \infty$
- Let $n = \frac{k}{r}$, then $k = nr$

- $B = A\left(1 + \frac{1}{n}\right)^{nt}$
- When $k \to \infty$, so does $n$
- $B = A\left[\left(1 + \frac{1}{n}\right)^n\right]^{rt}$
- Recall $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

- $B = Ae^{rt}$ for continuous compounding.

**Example**: Same situation as Example One, except interest is compounded continuously.

- $1000e^{(0.05)(2)} = 1000e^{0.1} \approx 1197.22$

**THE BOTTOM LINE**: Whatever way you decide to think of “e” (theoretical or practical), whenever growth is compounded continuously use $B = Ae^{rt}$ and whenever growth is compounded non-continuously use $B = A\left(1 + \frac{r}{k}\right)^{kt}$. 
Real world example of things that are continuously compounded:
1) Interest (as we have seen)
2) Population
3) Nuclear Decay (growth is negative)

Practical uses include (Discounting and Present Value)
Carbon Dating
Sometimes B is referred to as the "future value" of a dollar amount in

\[ B = A e^{rt} \]  or  \[ B = A (1 + r)^t \]

If we are given a future value and want to find the present value of something, we can also use these formulas to solve for A instead of B:

\[ A = \frac{B}{e^{rt}} \]  or  \[ A = \frac{B}{(1 + r)^t} \]

Example: someone says they will pay you $500 they owe you in a lump sum in exactly two years. If the interest rate remains constant at 6%, and is continuously compounded, what is the present value of that money?

\[ A = 500 e^{-0.06(2)} = \$ 443.46 \]
Now - we turn to logs

Logs and exponents are related, in that logs sort of reverse exponentiation.

Logs come with bases and mean the following:

Numbers are expressed in base 10, meaning

- \(10^0 = 1\)
- \(10^1 = 10\)
- \(10^2 = 100\)
- \(10^3 = 1000\)

Base 2 is
- \(2^0 = 1\)
- \(2^1 = 2\)
- \(2^2 = 4\)
- \(2^3 = 8\)

\[ \log_2 4 = 2 \] meaning 2 raised to the what is four? \( A = 2 \)

\[ \log_3 9 = 2 \]

\[ \log_{10} 1000 = 3 \]

\[ \log_{10} 10 = -2 \]

In math, the log 10 is standard, and we usually only see log 10 = 1 or similar. When you see log, assume it is base 10 unless otherwise noted.

We are also interested in the loge \( e^4 \)

\[ \log_e e^4 = 4 \]

The log base e is referred to in mathematics as the natural log, or ln.
Properties of logs:

1. You can NEVER take the log / ln of a negative number. Why?
   \[ \log_{10} (-10) = ? \quad 10^{x} = -10 \]
   \[ \ln (-2.71) = -2.71 \quad e^{x} = -2.71 \]

2. When I raise a positive number to ANY exponent, I never get a negative number.

3. \( \ln e^{x} = x \) \hspace{1cm} \text{Ex. } \ln e^{4} = 4 \)
4. \( e^{\ln x} = x \) \hspace{1cm} \ln x = b \text{ if } x = e^{b} \\]
5. \( e^{\ln x} = x \) \hspace{1cm} \text{Ex. } e^{\ln (2x)} = 2x \)

Other log rules:

6. \( \ln (u \cdot v) = \ln u + \ln v \)
   \[ \text{Example: } \ln (e^{4} \cdot e^{5}) = \ln (e^{9}) = 9 \]
   \[ \ln e^{4} + \ln e^{5} = 4 \cdot 5 = 20 \]

7. \( \ln \left( \frac{u}{v} \right) = \ln u - \ln v \)
   \[ \ln \left( \frac{e^{4}}{e^{5}} \right) = \ln e^{2} = 2 \]
   \[ \ln e^{4} - \ln e^{5} = 4 - 2 = 2 \]

8. \( \ln (e^{x}) = x \ln e = 2 \ln e = 2 \)
   \[ \text{This is the most important rule.} \]
That is one of the most important rules because of what it does to exponents — it makes them fall down.

These rules don't just apply to e — they apply to all variables and numbers, since afterall, e is just a number (like π).

Example: \( \ln(3x) = \ln 3 + \ln x \)
\[
\ln(\sqrt{ab}) = \ln(\sqrt[4]{a^2 b^3}) \\
= \frac{1}{2} \ln a + \frac{1}{2} \ln b \\
= \frac{1}{2}(\ln a + \ln b)
\]
\( \ln x^4 = 4 \ln x \)
\( \ln 3^2 = 2 \ln 3 = 2.197 \)
\( \ln 9 = \frac{\ln 3x}{2} \rightarrow 2.197 \)
\( \ln 3x = \ln 3 + \ln x \)

\( \ln e^t = 6t \ln e = 6t \)

Let's put this to use:

How quickly will money double if it is invested at 6% and compounded continuously?

\[ A = P e^{rt} \]
\[ 2A = 1000 e^{0.06t} \rightarrow \text{Solve for } t \]
\[ 2 = e^{0.06t} \]
\[ \ln 2 = \ln e^{0.06t} \rightarrow \ln e = 0.06t \]
\[ \ln 2 = 0.06t \]

\[ t = \frac{\ln 2}{0.06} \approx 11.60 \text{ years} \]
Problem #33 (P. 326)

\[ 9000 = 6000e^{20r} \]
\[ 1.5 = e^{20r} \]
\[ \ln 1.5 = 20r \ln e \]
\[ \frac{\ln 1.5}{20} = r \]
\[ r = 0.0203 \]

HW #3(b)

P. 309 (1, 5, 14, 15, 16, 20, 24, 28, 32, 33)
P. 325 (9, 14, 16p, 28, 30, 37, 38)

\[ ^{37}\text{Ar} \text{ has a half-life of } 5,730 \text{ years} \]